A Linear Operator Inequalities/LMI Approach to Stability and Control of Distributed Parameter Systems

Emilia FRIDMAN*

Electrical Engineering, Tel Aviv University, Israel

April 3, 2011

Plan

1. Introduction

2. Exponential stability of linear time-delay systems in the Hilbert space
   - Linear Operator Inequalities for Exp. Stability in a Hilbert Space
   - LMIs for the Delay Heat Equation
   - LMIs for the Delay Wave Equation
   - Conclusions

3. Bounds on the response of a drilling pipe model
   - Motivation: *improved drilling towards Leviathan gas discovery*
   - Drilling system model: a 1-d wave equation
   - Ultimate boundedness

- Two main approaches are usually used for stability and control of infinite-dimensional systems:
    1. the analysis and control of the abstract infinite-dimensional system (e.g. in the Hilbert space) with the corresponding conclusions for specific systems;
    2. the direct approach to a specific system.
- In this talk both approaches to Lyapunov-based analysis will be presented:
    1. the Linear Operator Inequalities (LOIs) for the stability of linear time-delay systems in a Hilbert space.
       [**Fridman & Orlov**, Aut09a]
       Thanks to *J.-P. RICHARD* for our visits to Ecole Centrale de Lille.
    2. the direct Lyapunov approach to analysis of 1-d wave eq.
       [**Fridman & Orlov**, Aut09b], [**Fridman, S. Mondie, B. Saldivar** , IMA J.]

Plan

- Delays may be a source of instability. However, they may have also a stabilizing effect.
- In the case of distributed parameter systems, arbitrarily small delays in the feedback may **destabilize** the system [**Datko**, SICON 88], [**Logemann** *et al.*, SICON 96], [**Nicaise & Pignotti**, SICON 06].
- Thus, the wave eq. non-robust w.r.t. delay [**Wang, Guo & Krstic**, SICON 11]:

$$\begin{aligned} &z_{tt}(\xi,t) = z_{\xi\xi}(\xi,t), \quad \xi \in (0,1), \\ &z(0,t) = 0, \quad z_{\xi}(1,t) = kz_t(1,t-h) \end{aligned}$$

  - is stable for $h = 0$ and $k = 1$ (all solutions are zero for $t \geq 2$), unstable for all small enough h and $k = 1$ [**Datko**, TAC 97]
  - stable for $h = 2$ iff $k \in (0,1)$, unstable for arbitrary small perturbations of $h = 2$,
  - for $h = 2,4,6,8$ (even multiples of the wave propagation) stable for some $k > 0$,
  - for $h = 1,3,5,0.5$ unstable $\forall k$.

- The stability analysis of PDEs with delay is essentially more complicated than of ODEs.
- There are only a few works on Lyapunov-based technique for PDEs with delay. The 2nd Lyapunov method was extended to abstract nonlinear time-delay systems in the Banach spaces in Wang (1994a, JMAA) and applied to scalar heat and scalar wave equations with constant delays and with the Dirichlet boundary conditions in Wang (1994b, JMAA), Wang(2006, JMAA).
- Stability and instability conditions for wave delay equations were found in (Nicaise & Pignotti, SIAM 2006).

- In **(E. Fridman & Y. Orlov, Aut 09)** exp. stability of general distributed parameter systems are derived for linear systems, where a *bounded operator acts on the delayed state*. The system delay is admitted to be *unknown and time-varying*.
- Sufficient *exp. stability* conditions are derived in the form of Linear Operator Inequalities (LOIs), where the decision variables are operators in the Hilbert space.
  General methods for solving LOI have not been developed yet. Some *finite dimensional approximations* were considered in Ikeda, Azuma & Uchida (2001).
- Being applied to a *heat/wave* equation these conditions are represented in terms of standard finite-dimensional LMIs that guarantee the stability of the 1-st/2-nd order delay-differential eqs. This *reduction of LOIs to finite-dimensional LMIs is tight*: the stability of the latter delay-differential eqs is necessary for the stability of the heat/wave eqs.

Problem Statement

- $$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geq t_0 \tag{1}$$

  where $x(t) \in \mathcal{H}$, $\mathcal{H}$ is a Hilbert space,
  delay $\tau(t)$ is a piecewise continuous function

  $$\inf_t \tau(t) > 0, \ \sup_t \tau(t) \leq h, \ h > 0 \tag{2}$$

  A_1 is a linear bounded operator,
  A is an infinitesimal operator, generating a strongly continuous semigroup $T(t)$, the domain $\mathcal{D}(A)$ is dense in $\mathcal{H}$.
- Throughout, solutions of such a system are defined in the Caratheodory sense: (1) is required to hold almost everywhere.
- Let the initial conditions $x^{t_0} = \varphi(\theta), \ \theta \in [-h, 0], \phi \in W$ be given in the space $W = C([-h, 0], \mathcal{D}(A)) \cap C^1([-h, 0], \mathcal{H})$.

Let the initial conditions

$$x^{t_0} \triangleq x(t_0+\theta) = \varphi(\theta),\ \theta \in [-h,0], \phi \in W$$

be given in the space $W = C([-h,0],\mathcal{D}(A)) \cap C^1([-h,0],\mathcal{H})$.

- Under the assumption

$$\inf_t \tau(t) = h_0 > 0,\ \sup_t \tau(t) \le h,\ h > 0 \tag{3}$$

  we have $\tau(t) \in [h_0, h]$.
  The above initial-value problem is *well-posed* on $[t_0, \infty)$ and its solutions can be found as mild solutions of

$$\begin{aligned} &x(t) = T(t-t_0)x(t_0) \\ &+ \int_{t_0}^{t} T(t-s)A_1 x(s-\tau(s))ds,\ t \ge t_0. \end{aligned} \tag{4}$$

- In this talk we will consider 2 main examples:
  1) heat (parabolic eq); 2) wave (hyperbolic eq).
- **Example 1**: heat eq.

$$z_t(\xi,t) = az_{\xi\xi}(\xi,t) - a_1 z(\xi,t-\tau(t)),\ t \geq t_0,\ 0 \leq \xi \leq \pi \qquad (5)$$

with constants $a > 0$ and a_1 and with the Dirichlet b. c.

$$z(0,t) = z(\pi,t) = 0,\ t \geq t_0. \qquad (6)$$

a is the heat conduction coefficient,
a_1 is the coefficient of the heat exchange with the surroundings
$z(\xi,t)$ is the temperature of the rod
The above system describes the propagation of heat in a homogeneous 1-d rod with a fixed temperature at the ends in the case of the delayed (possibly, due to actuation) heat exchange.

- Heat eq. can be rewritten as

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geq t_0$$

$\mathcal{H} = L_2(0, \pi)$, $A = a\frac{\partial^2}{\partial \xi^2}$ with the dense domain

$$\mathcal{D}(\frac{\partial^2}{\partial \xi^2}) = \{z \in W^{2,2}([0, \pi], \mathbf{R}) : z(0) = z(\pi) = 0\},$$

and with the **bounded operator** $A_1 = -a_1$.

- A **generates a strongly continuous semigroup** (see, e.g., Curtain & Zwart (1995) for details).

Example 2

:

- Wave equation

$$\begin{array}{l} z_{tt}(\xi,t) = az_{\xi\xi} - \mu_0 z_t(\xi,t) - a_0 z(\xi,t) \\ -a_1 z(\xi,t-\tau(t)), \quad t \geq 0,\ 0 \leq \xi \leq \pi, \\ z(0,t) = z(\pi,t) = 0,\ t \geq t_0. \end{array} \tag{7}$$

Eqs (7) describe the **oscillations of a homogeneous string with fixed ends** in the case of the *delayed stiffness restoration*.

- Introduce the operators

$$A = \begin{bmatrix} 0 & 1 \\ a\frac{\partial^2}{\partial\xi^2} - a_0 & -\mu_0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ -a_1 & 0 \end{bmatrix}$$

$$\mathcal{D}(\frac{\partial^2}{\partial\xi^2}) = \{z \in W^{2,2}([0,\pi],\mathbf{R}) : z(0) = z(\pi) = 0\},$$

- Then (7) can be represented as

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geq t_0$$

in $\mathcal{H} = L_2(0,\pi) \times L_2(0,\pi)$ with the infinitesimal operator A,

- possessing the domain $\mathcal{D}(A) = \mathcal{D}(\frac{\partial^2}{\partial \xi^2}) \times L_2(0,\pi)$ and generating a strongly continuous semigroup (see, e.g., Curtain & Zwart (1995)).

Our aim is to derive exp. stability criteria for (1), (2). The stability concept under study is based on the initial data norm in the space

$$W = C([-h,0],\mathcal{D}(A)) \cap C^1([-h,0],\mathcal{H})$$

defined as

$$\|\phi\|_W = |A\phi(0)| + \|\phi\|_{C^1([-h,0],\mathcal{H})} \tag{8}$$

Suppose $x(t,t_0,\phi),\ t \geq t_0$ denotes a solution of (1) with $x^{t_0} = \phi$. System (1) is said to be exponentially stable with a decay rate $\delta > 0$ if $\exists K \geq 1$:

$$|x(t,t_0,\phi)|^2 \leq Ke^{-2\delta(t-t_0)}\|\phi\|_W^2 \quad \forall t \geq t_0. \tag{9}$$

- Given a continuous functional $\mathbf{V}:\mathbf{R}\times\mathbf{W}\times\mathbf{C}([-\mathbf{h},\mathbf{0}],\mathcal{H})\to\mathbf{R}$, $\dot{V}$ along (1) is defined as follows:

$$\dot{V}(t,\phi,\dot{\phi}) = \limsup_{s\to 0^+}\frac{1}{s}[V(t+s,x^{t+s}(t,\phi),\dot{x}^{t+s}(t,\phi)) - V(t,\phi,\dot{\phi})].$$

- **Lemma** Let $\exists\delta,\ \beta,\ \gamma$ and a continuous functional

$$V:\mathbf{R}\times W\times C([-h,0],\mathcal{H})\to\mathbf{R}$$

such that the function $\bar{V}(t)=V(t,x^t,\dot{x}^t)$ is absolutely continuous for x^t, satisfying (1), and

$$\begin{array}{l}\beta|\phi(0)|^2\le V(t,\phi,\dot{\phi})\le\gamma\|\phi\|_W^2,\\ \dot{V}(t,\phi,\dot{\phi})+2\delta V(t,\phi,\dot{\phi})\le 0.\end{array}\qquad(10)$$

Then (1) is exp. stable with the decay rate δ and with $K=\frac{\gamma}{\beta}$.

Notation:
Given a linear operator $\Phi : \mathcal{H} \to \mathcal{H}$ with a dense domain $\mathcal{D}(\Phi) \subset \mathcal{H}$, the notation Φ^* stands for the adjoint operator. Such an operator Φ is strictly positive definite, i.e., $\Phi > 0$, iff it is self-adjoint, i.e. $\Phi = \Phi^*$ and $\exists \beta > 0$ such that

$$\langle x, \Phi x \rangle \geq \beta \langle x, x \rangle, \forall x \in \mathcal{D}(\Phi)$$

$\Phi \geq 0$ means that $\langle x, \Phi x \rangle \geq 0$ for all $x \in \mathcal{D}(\Phi)$.

In a Hilbert space $\mathcal{D}(A)$, consider

$$V(t,x^t,\dot{x}^t) = \langle x(t), Px(t)\rangle + \int_{t-h}^{t} e^{2\delta(s-t)}\langle x(s), Sx(s)\rangle ds$$
$$+h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\delta(s-t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds d\theta + \int_{t-\tau(t)}^{t} e^{2\delta(s-t)}\langle x(s), Qx(s)\rangle ds$$

$\mathbf{P}:\mathcal{D}(\mathbf{A}) \to \mathcal{H}$ is a linear operator, $P>0$,
$R,Q,S \in \mathcal{L}(\mathcal{H})$, $R,Q,S \geq 0$
$\forall x \in D(A)$ and some positive $\gamma_P, \gamma_Q, \gamma_S, \gamma_R$

$$\begin{array}{l}\langle x, Px\rangle \leq \gamma_P[\langle x,x\rangle + \langle Ax, Ax\rangle], \quad \langle x, Qx\rangle \leq \gamma_Q\langle x,x\rangle, \\ \langle x, Rx\rangle \leq \gamma_R\langle x,x\rangle, \ \langle x, Sx\rangle \leq \gamma_S\langle x,x\rangle\end{array} \tag{11}$$

By using **Cauchy-Schwartz (Jensen's) inequality**, we obtain conditions in 2 forms:
1) by substituting the right side of (1) for $\dot{x}(t)$;
2) by using descriptor approach (*Fridman SCL 2001*):

Theorem 1 (1) is exp. stable with the decay rate δ if LOI is feasible

$$\Phi_h = \begin{bmatrix} \Phi_{11} & 0 & PA_1 \\ 0 & 0 & 0 \\ A_1^*P & 0 & 0 \end{bmatrix} + h^2 \begin{bmatrix} A^*RA & 0 & A^*RA_1 \\ 0 & 0 & 0 \\ A_1^*RA & 0 & A_1^*RA_1 \end{bmatrix}$$
$$-e^{-2\delta h} \begin{bmatrix} R & 0 & -R \\ 0 & (S+R) & -R \\ -R & -R & 2R+(1-d)Q \end{bmatrix} \leq 0,$$

where $\Phi_h : \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A) \to \mathcal{H} \times \mathcal{H} \times \mathcal{H}$ and where

$$\Phi_{11} = A^*P + PA + 2\delta P + Q + S. \qquad (12)$$

Differently from the finite dimensional case, the feasibility of the strict LOIs for $h = 0$ (or $\delta = 0$) does not necessarily imply the feasibility of these LOIs for small enough h (δ) because h^2 (δ) is multiplied by the operator, which may be unbounded.

Theorem 1 gives delay-dependent conditions (h-dependent) even for $\delta \to 0$. For $S = R = 0$ we obtain the following "quasi delay-independent" conditions, which coincide for ODE with (Mondie & Kharitonov, TAC 2005):

Corollary Given $\delta > 0$, (1) is exp. stable with the decay rate δ for all delays with $\dot{\tau}(t) \leq d < 1$ if $\exists\ P > 0$ and $Q \geq 0$ subject to (11) such that the LOI

$$\begin{bmatrix} (A+\delta)^*\mathbf{P} + \mathbf{P}(A+\delta)+Q & PA_1 \\ A_1^*P & -(1-d)Qe^{-2\delta h} \end{bmatrix} \leq 0 \qquad (13)$$

holds in $\mathcal{D}(A) \times \mathcal{D}(A) \to \mathcal{H} \times \mathcal{H}$. The inequality (9) is satisfied with $K = \max\{\gamma_P, h\gamma_Q\}/\beta$.

It may be difficult to verify the feasibility of LOI 1, if **the operator that multiplies h^2 (and depends on A) in Φ_h is unbounded**. To avoid this, we will derive the *2-nd form* of LOI by the *descriptor method* (Fridman, SCL 2001), where the right-hand sides of the expressions

$$\begin{aligned} 0 &= 2\langle x(t), P_2^*[Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)]\rangle, \\ 0 &= 2\langle \dot{x}(t), P_3^*[Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)]\rangle \end{aligned} \tag{14}$$

with some $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ are added into the right-hand side of $\dot{V}$.

LOI 2 via descriptor method

$$\begin{bmatrix} \Phi_{d11} & \Phi_{d12} & 0 & P_2^* A_1 + Re^{-2\delta h} \\ * & \Phi_{d22} & 0 & P_3^* A_1 \\ * & * & -(S+R)e^{-2\delta h} & Re^{-2\delta h} \\ * & * & * & -[2R+(1-d)Q]e^{-2\delta h} \end{bmatrix} \leq 0 \tag{15}$$

holds, where

$$\begin{aligned} \Phi_{d11} &= A^* P_2 + P_2^* A + 2\delta P + Q + S - Re^{-2\delta h}, \\ \Phi_{d12} &= P - P_2^* + A^* P_3, \\ \Phi_{d22} &= -P_3 - P_3^* + h^2 R. \end{aligned} \tag{16}$$

$$z_t(\xi,t) = az_{\xi\xi}(\xi,t) - a_0 z(\xi,t) - a_1 z(\xi,t-\tau(t)),$$
$$t \geq t_0,\ 0 \leq \xi \leq \pi \tag{17}$$

with constant $a > 0$, a_0, a_1 and with the Dirichlet boundary conditions

$$z(0,t) = z(\pi,t) = 0,\ t \geq t_0. \tag{18}$$

Here we **apply the descriptor method LOIs**. The boundary-value problem (17), (18) can be rewritten as (1) in the Hilbert space $\mathcal{H} = L_2(0,\pi)$ with $A = a\frac{\partial^2}{\partial\xi^2} - a_0$ with the dense domain

$$\mathcal{D}(\frac{\partial^2}{\partial\xi^2}) = \{z \in W^{2,2}([0,\pi],\mathbf{R}) : z(0) = z(\pi) = 0\}, \tag{19}$$

and with the bounded operator $A_1 = -a_1$.
A generates a strongly continuous semigroup

Delay-independent conditions are derived by using

$$V = p\int_0^\pi z^2(\xi,t)d\xi + q\int_{t-\tau(t)}^t \int_0^\pi e^{2\delta(s-t)} z^2(\xi,s)d\xi ds \tag{20}$$

with some constants $p > 0$ and $q > 0$. Then $P = p, \quad Q = q$.
Integrating by parts and taking into account boundary conditions, we find

$$\langle x, (A^*P + PA)x\rangle = 2a\int_0^\pi pzz_{\xi\xi}d\xi = -2a\int_0^\pi pz_\xi^2 d\xi \leq -2a\int_0^\pi pz^2 d\xi$$

for $x \in \mathcal{D}(A)$, where the last inequality follows from the Wirtinger's Inequality [Hardy et al., 88]: Let $z \in W^{1,2}([a,b], R)$ be a scalar function with $z(a) = z(b) = 0$. Then

$$\int_a^b z^2(\xi)d\xi \leq \frac{(b-a)^2}{\pi^2}\int_a^b (z'(\xi))^2 d\xi. \tag{21}$$

We thus obtain that the LOI is satisfied provided that the following LMI

$$\begin{bmatrix} q-2(a+a_0)p & -a_1p \\ -a_1p & -(1-d)qe^{-2\delta h} \end{bmatrix} < 0 \tag{22}$$

is feasible.
LMI (22) with $\delta = 0$ is feasible iff $\mathbf{a}+\mathbf{a_0} > \mathbf{0}, \quad \mathbf{a_1^2} < (\mathbf{a}+\mathbf{a_0})^{\mathbf{2}}(\mathbf{1}-\mathbf{d})$.

Delay-Dependent Conditions

Choose the LKF of the form

$$V(t, z^t, z_s^t) = (p_1 - p_3 a) \int_0^\pi z^2(\xi, t) d\xi + p_3 a \int_0^\pi z_\xi^2(\xi, t) d\xi$$

$$+ \int_0^\pi \Big[r \int_{-h}^0 \int_{t+\theta}^t e^{2\delta(s-t)} z_s^2(\xi, s) ds d\theta$$

$$+ s \int_{t-h}^t e^{2\delta(s-t)} z^2(\xi, s) ds + q \int_{t-\tau(t)}^t e^{2\delta(s-t)} z^2(\xi, s) ds \Big] d\xi$$

with $p_1 > 0, p_3 > 0, s > 0, r > 0$ and $q \geq 0$.
Then the operators in LOI take the form

$$P = -p_3 (a \frac{\partial^2}{\partial \xi^2} + a) + p_1, \; R = r, \; Q = q, \; S = s,$$

$$P_3 = p_3, \quad P_2 = p_2 > 0, \; p_2 - \delta p_3 \geq 0$$

Integrating by parts and applying Wirtinger's Inequality

$$\int_a^b z^2(\xi)d\xi \leq \frac{(b-a)^2}{\pi^2}\int_a^b (z'(\xi))^2 d\xi \qquad (23)$$

we show that $P > 0$ and that the LOI of Th.2 is feasible if the following LMI is feasible

$$\begin{bmatrix} \phi_{11} & \phi_{12} & 0 & \phi_{14} \\ * & -2p_3 + h^2 r & 0 & -p_3 a_1 \\ * & * & -(s+r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & * & \phi_{44} \end{bmatrix} < 0,$$

$$\phi_{11} = -2(a+a_0)p_2 + 2\delta p_1 + q + s - re^{-2\delta h},$$

$$\phi_{12} = p_1 - p_2 - (a+a_0)p_3, \ \phi_{14} = -p_2 a_1 + re^{-2\delta h},$$
$$\phi_{44} = -[2r + (1-d)q]e^{-2\delta h}$$

Remark

The same LMIs guarantee the exp. stability of the scalar equation

$$\dot{y}(t) + (a + a_0)y(t) + a_1 y(t - \tau(t)) = 0 \tag{24}$$

Eq. (24) corresponds to the first modal dynamics (with $k = 1$) in the modal representation of the Dirichlet b. v. problem for the heat equation

$$y_k(t) + (a + a_0)k^2 y_k(t) + a_1 y_k(t - \tau(t)) = 0,$$

$k = 1, 2, \ldots$ projected on the eigenfunctions of the operator $\frac{\partial^2}{\partial \xi^2}$ (this operator has eigenvalues $-k^2$).
The stability of the heat eq. implies the stability of ODE (24).
Thus the reduction of infinite-dimensional LOI to finite-dimensional LMIs is tight: the stability of (24) is necessary for the stability of the heat eq.

Remark

- The above LOIs (LMIs) are affine in the system operators (coefficients). Consider now the systems under question with the uncertain operators (coefficients) from the uncertain polytope, given by M vertices. By the arguments of (Boyd *et al.* 1994), the uncertain systems are exp. stable if the corresponding LOIs (LMIs) in the vertices are feasible.

Example Consider the controlled heat eq.

$$\begin{array}{c} z_t(\xi,t) = z_{\xi\xi}(\xi,t) + rz(\xi,t) + u, \\ z(0,t) = z(l,t) = 0, \end{array} \tag{25}$$

where $\xi \in (0,l)$, $t > 0$ and where r is uncertain parameter satisfying $|r| \leq \beta$.

- It was shown in (Rebiai & Zinober, IJC 1993) that for $l = 1$
  $u = -\gamma z(\xi, t),\ \gamma > (\frac{\beta}{2\pi})^2$ as. stabilizes (25).
  By our method $u = -\gamma z(\xi, t),\ \gamma > \beta - \pi^2$ Since $\beta - \pi^2 \leq (\frac{\beta}{2\pi})^2$,
  our method guarantees exp. stabilization via a lower gain.
- Consider next $l = \pi,\ \beta = 0.1$ and the feedback $u = -z(\xi, t - \tau(t))$
  with the uncertain delay $\tau(t) \in [0, h],\ \dot{\tau} \leq d < 1$
  This is a **polytopic system reached by choosing** $\mathbf{r} = \pm \mathbf{0.1}$. We verify the feasibility of delay-dependent LMIs in 2 vertices: $r = \pm 0.1$.
  We use LMI toolbox of Matlab and find the maximum values of h for which the system remains as. stable:

$$d = 0.5,\ h = 2.04;\ unknown\ d,\ h = 1.34.$$

The latter results correspond also to the stability of

$$\dot{y} = (-1 + r) y(t) - y(t - \tau(t)),\ |r| \leq 0.1$$

Consider the wave equation

$$
\begin{aligned}
&z_{tt}(\xi,t) = a z_{\xi\xi} - \mu_0 z_t(\xi,t) - \mu_1 z_t(\xi,t-\tau(t)) \\
&-a_0 z(\xi,t) - a_1 z(\xi,t-\tau(t)), \quad t \geq 0,\ 0 \leq \xi \leq \pi
\end{aligned}
\tag{26}
$$

with the Dirichlet boundary condition (18).
Introduce the operators

$$
A = \begin{bmatrix} 0 & 1 \\ a\frac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -a_1 & -\mu_1 \end{bmatrix}
\tag{27}
$$

where the domain $\mathcal{D}(\frac{\partial^2}{\partial \xi^2})$ is determined by (19). Then (18), (26) can be represented as (1) in the Hilbert space $\mathcal{H} = L_2(0,\pi) \times L_2(0,\pi)$ with the infinitesimal operator A, possessing the domain $\mathcal{D}(A) = \mathcal{D}(\frac{\partial^2}{\partial \xi^2}) \times L_2(0,\pi)$ and generating a strongly continuous semigroup (see, e.g., Curtain & Zwart (1995)).

Delay-Dependent Conds: $\mu_1 = 0$ We apply LOI of Theorem 1. Since the delay appears only in u, we choose V as follows:

$$V = ap_3 \int_0^\pi z_\xi^2(\xi,t)d\xi + \int_0^\pi [z(\xi,t)\ z_t(\xi,t)] P_0 \begin{bmatrix} z(\xi,t) \\ z_t(\xi,t) \end{bmatrix} d\xi$$
$$+ \int_0^\pi \Big[hr \int_{-h}^0 \int_{t+\theta}^t z_t^2(\xi,s) e^{2\delta(s-t)} ds d\theta + s \int_{t-h}^t z^2(\xi,s) e^{2\delta(s-t)} ds$$
$$+ q \int_{t-\tau}^t z^2(\xi,s) e^{2\delta(s-t)} ds \Big] d\xi,$$
$$P_0 = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}, \ P_w = \begin{bmatrix} ap_3 + p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0.$$

where $r > 0,\ s > 0,\ q \geq 0$. Then the operators P, Q, R in LOI are given by

$$P = \begin{bmatrix} -ap_3 \frac{\partial^2}{\partial \xi^2} + p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0,\ Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \geq 0,$$
$$R = diag\{r, 0\} \geq 0,\ S = diag\{s, 0\} \geq 0.$$

Denote

$$C_\delta = \begin{bmatrix} \delta & 1 \\ -a - a_0 & -\mu_0 + \delta \end{bmatrix}. \qquad (28)$$

LOI is feasible if the following LMIs are satisfied:

$$p_2 \geq p_3\delta,$$
$$\begin{bmatrix} \phi_w & 0 & P_w \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} re^{-2\delta h} \\ 0 \end{bmatrix} \\ * & -(s+r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & -(2r + (1-d)q)e^{-2\delta h} \end{bmatrix} < 0, \qquad (29)$$

where

$$\phi_w = C_\delta^T P_w + P_w C_\delta + diag\{q + s - re^{-2\delta h}, h^2 r\}.$$

The same LMIs appear to guarantee the stability of ODE with delay

$$\dot{\bar{z}}(t) = C_0\bar{z}(t) + A_1\bar{z}(t - \tau(t)), \quad \bar{z}(t) \in \mathbf{R}^2.$$

This ODE governs the first modal dynamics of the modal representation of the Dirichlet boundary-value problem.

Example

- Consider the controlled wave equation

$$z_{tt}(\xi,t) = 0.1 z_{\xi\xi}(\xi,t) - 2z_t(\xi,t) + u, \qquad (30)$$

- with boundary condition (18),
  $t \geq t_0,\ 0 \leq \xi \leq \pi, 0 \leq \tau \leq h, \dot{\tau} \leq d < 1.$
- Applying LMI to the open-loop system we find that (30) with $u = 0$ is **exp. stable** with the decay rate $\delta = 0.05$.
- Considering next a delayed feedback
  $u = -z(\xi, t - \tau(t))$
- and verifying the LMI, we find that the closed-loop system is exp. stable with a greater decay rate $\delta = 0.8$ for all $0 \leq \tau(t) \leq 0.31$.

Delay-Ind. Conditions are derived for $\mu_1 \neq 0$.

- **Corollary from delay-ind. conds** In a particular case where $a = 1$, $a_0 = a_1 = 0$, the Dirichlet boundary-value problem for the wave eq. is exp. stable for all $\dot{\tau} \leq d < 1$ if $\mu_1^2 < (1-d)\mu_0^2$
- **Remark**. The condition $0 \leq \mu_1 < \mu_0$ for the stability of the wave eq. with *constant delay* and $a = 1$, $a_0 = a_1 = 0$ was obtained by *Nicaise & Pignotti (SIAM 2006)*, where it was shown that if $\mu_1 \geq \mu_0$, there exists a sequence of arbitrary small delays that destabilize the system.

Thanks to **Jack Hale** for encouragement on Partial Functional Dif. Eqs and to **Chris Byrnes** for discussions on
Output Regulation for Distributed Parameter Systems with Delays

- A general framework is given for exp. stability of linear distributed parameter systems in a Hilbert space with a bounded operator acting on the delayed state.
- Stability conditions are derived in terms of LOIs in the Hilbert space.
- In the case of a heat/wave scalar equation with the Dirichlet boundary conditions, these LOIs are reduced to finite-dimensional LMIs by applying new Lyapunov functionals.
- The simplicity and the tightness of the results are the advantages of the new method.
- As it happened with LMIs in the finite dimensional case, LOIs are expected to provide effective tools for robust control of distributed parameter systems.

Plan

1. Introduction

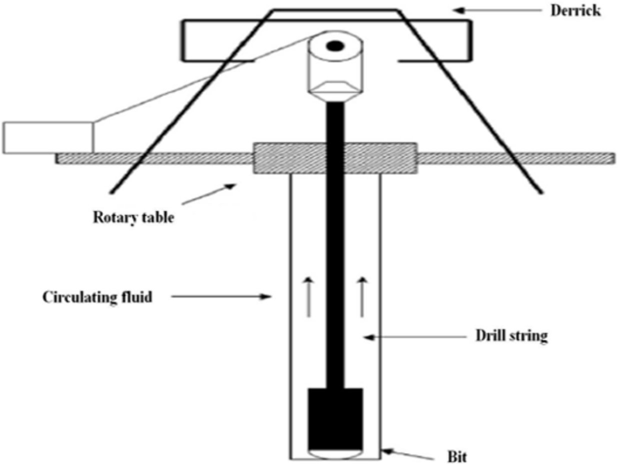
2. Exponential stability of linear time-delay systems in the Hilbert space
   - Linear Operator Inequalities for Exp. Stability in a Hilbert Space
   - LMIs for the Delay Heat Equation
   - LMIs for the Delay Wave Equation
   - Conclusions

3. Bounds on the response of a drilling pipe model
   - Motivation: *improved drilling towards Leviathan gas discovery*
   - Drilling system model: a 1-d wave equation
   - Ultimate boundedness

Joint with Sabine Mondie and Belem Saldivar (Cinvestav, Mexico) (IMA J. 2010)

1. Motivation: *improved drilling towards Leviathan gas discovery*
2. Drilling system model
3. Ultimate boundedness:
    - difference equation approach
    - wave equation analysis
4. Numerical examples
5. Concluding remarks

Motivation: *improved drilling towards Leviathan gas discovery*



A sketch of a simplified drillstring system is shown on Fig. 1.

Drilling system model

N. Challamel, *Rock destruction effect on the stability of a drilling structure*, Journal of sound and vibration, 233 (2), 235-254, 2000.

$$\frac{GJ}{L^2}\frac{\partial^2 z}{\partial \sigma^2}(\sigma,t) - I\frac{\partial^2 z}{\partial t^2}(\sigma,t) - \beta\frac{\partial z}{\partial t}(\sigma,t) = 0, \qquad \sigma \in [0,1],$$

$$z(0,t) = 0;$$
$$\tfrac{GJ}{L}\tfrac{\partial z}{\partial \sigma}(1,t) + I_B\tfrac{\partial^2 z}{\partial t^2}(1,t) = -T'(\Omega + \theta z_t)\tfrac{\partial z}{\partial t}(1,t) + w(t).$$

- $z(\sigma,t)$ is the angle of rotation,
- T is the torque on the bit,
- I_B is a lumped inertia (the assembly at the bottom hole),
- $\beta \geq 0$ damping (viscous and structural),
- I is the inertia, G is the shear modulus, J is the geometrical moment of inertia.

- The presence of a bounded additive noise signal $w(t)$ is considered at the bottom of the drillstring in order to account for external disturbances and modeling errors

$$|w(t)| \leq \overline{w},\ t \in (0, \infty).$$

- We will study ultimate boundedness of the solution.

The initial conditions are:

$$\begin{array}{l} z(\sigma,0) = \zeta(\sigma), \quad z_\sigma(\sigma,0) = \dot{\zeta}(\sigma) \in L_2(0,1), \\ z_t(\sigma,0) = \zeta_1(\sigma) \in L_2(0,1). \end{array} \tag{31}$$

When the damping and the lumped inertia are negligible ($\beta = I_B = 0$) the model reduces to:

$$\frac{\partial^2 z}{\partial t^2}(\sigma,t) = a\frac{\partial^2 z}{\partial \sigma^2}(\sigma,t), \qquad \sigma \in [0,1],\ t \geq 0 \tag{32}$$

$$z(0,t) = 0; \qquad \frac{\partial z}{\partial \sigma}(1,t) = -k\frac{\partial z}{\partial t}(1,t) + rw(t) \tag{33}$$

where $a = \frac{GJ}{IL^2}$, $k = \frac{LT'}{GJ}$, $r = \frac{L}{GJ} \in \mathbb{R}$.

A differential-difference equation approach

The general solution of the unidimensional wave equation is:

$$z(\sigma,t)=\phi(t+s\sigma)+\psi(t-s\sigma),\ t\geq s,$$

where ϕ, $\psi\in C^1$ and $s=\sqrt{\frac{1}{a}}$.

The boundary conditions can be rewritten as:

$$z(0,t)=\phi(t)+\psi(t)=0,$$

$$\frac{\partial z(1,t)}{\partial\sigma}=s\dot{\phi}(t+s)-s\dot{\psi}(t-s)=-k[\dot{\phi}(t+s)+\dot{\psi}(t-s)]+rw(t).$$

A differential-difference equation approach

It follows from the above expressions that

$$\phi(t) = -\psi(t), \ t \geq 0.$$

We obtain the differential-difference equation for $t \geq s$:

$$\dot{\psi}(t+s) = -\frac{(s-k)}{(s+k)}\dot{\psi}(t-s) - \frac{r}{(s+k)}w(t), \ s = \sqrt{\frac{1}{a}}$$

with the initial condition

$$\dot{\psi}(t) = -0.5[\zeta_1(t/s) + \dot{\zeta}(t/s)/s], \ t \in [0, s],$$
$$\dot{\psi}(-t) = 0.5[\zeta_1(t/s) - \dot{\zeta}(t/s)/s], \ t \in [0, s].$$

A differential-difference equation approach

Notice that

$$\frac{\partial z}{\partial \sigma}(\sigma,t) = s\dot{\phi}(t+s\sigma) - s\dot{\psi}(t-s\sigma), \qquad \frac{\partial z}{\partial t}(\sigma,t) = \dot{\phi}(t+s\sigma) + \dot{\psi}(t-s\sigma).$$

We now summarize the above results:

Lemma

The solution of the boundary-value problem (32), (33) is ultimately bounded and for all initial conditions (31) it satisfies the inequalities

$$|z_\sigma(1,t)| \le s e^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{2s|c_1|}{1-e^{-\lambda}}\bar{w}, \quad t \ge 0,$$

$$|z_t(1,t)| \le e^{-\frac{\lambda}{s}t}[|\zeta_1(\xi/s)| + |\dot{\zeta}(\xi/s)|/s] + \frac{2|c_1|}{1-e^{-\lambda}}\bar{w} \quad t \ge 0.$$

A wave equation analysis

Under the assumption that the lumped inertia is negligible (i.e. $I_B = 0$) the model reduces to:

$$z_{tt}(\sigma,t) = a z_{\sigma\sigma}(\sigma,t) + d z_t(\sigma,t) \quad t \geq t_0,\ 0 \leq \sigma \leq 1$$

with the boundary conditions:

$$\begin{array}{l} z(0,t) = 0, \\ z_\sigma(1,t) = -k z_t(1,t) + r w(t),\ t \geq 0, \end{array}$$

where $a = \frac{GJ}{IL^2}$, $d = \frac{-\beta}{I}$, $r = \frac{L}{GJ}$ and $k = \frac{LT'}{GJ}$ with $0 < k_0 \leq k \leq k_1$.

$$\begin{array}{l} z(\sigma,0) = \zeta(\sigma), \quad z_\sigma(\sigma,0) = \dot{\zeta}(\sigma) \in L_2(0,1), \\ z_t(\sigma,0) = \zeta_1(\sigma) \in L_2(0,1). \end{array}$$

A wave equation analysis

Outline of the proof.

We look for an energy function V such that:

Lemma (Fridman & Dambrine, 2008)

Assume that $|w| \leq \bar{w}$. *Let* $V : [0, \infty) \to R^+$ *be an absolutely continuous function. If there exists* $\delta > 0$, $b > 0$ *such that the derivative of* V *satisfies almost everywhere the inequality*

$$\frac{d}{dt}V + 2\delta V - bw^2 \leq 0,$$

then it follows that

$$V(t) \leq e^{-2\delta(t-t_0)}V(t_0) + (1 - e^{-2\delta(t-t_0)})\frac{b}{2\delta}\bar{w}^2.$$

A wave equation analysis

Consider the energy function

$$V(z_\sigma(\cdot,t),z_t(\cdot,t)) = p[\int_0^1 a z_\sigma^2(\sigma,t)d\sigma + \int_0^1 z_t^2(\sigma,t)d\sigma]$$
$$+2\chi \int_0^1 \sigma z_\sigma(\sigma,t) z_t(\sigma,t)d\sigma$$

(Nicaise & Pignotti, 2006) with constants $p > 0$ and small enough χ.
Following LMI approach of (Fridman & Y. Orlov, Aut 09)

$$V(z_\xi(\cdot,t),z_t(\cdot,t)) \geq \int_0^1 [z_\xi \; z_t] \begin{bmatrix} a_1 p & \chi\xi \\ \chi\xi & p \end{bmatrix} [z_\xi \; z_t]^T d\xi$$
$$> \varepsilon \int_0^1 [z_\xi^2(\xi,t) + z_t^2(\xi,t)]d\xi \qquad (34)$$

for some $\varepsilon > 0$. The latter inequality holds if

$$\begin{bmatrix} a_1 p & \chi\xi \\ \chi\xi & p \end{bmatrix} > 0, \; \forall \xi \in [0,1] \Leftarrow \begin{bmatrix} a_1 p & \chi \\ \chi & p \end{bmatrix} > 0$$

A wave equation analysis

$$\begin{aligned}\frac{d}{dt}V &= 2p\int_0^1 a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t)d\sigma + 2p\int_0^1 z_t(\sigma,t)z_{tt}(\sigma,t)d\xi \\ &\quad +2\chi\frac{d}{dt}\left(\int_0^1 \sigma z_t z_\sigma d\sigma\right) \\ &= 2p\int_0^1 [a z_\sigma(\sigma,t) z_{t\sigma}(\sigma,t) + a z_t(\sigma,t) z_{\sigma\sigma}(\sigma,t)]d\sigma \\ &\quad +2pd\int_0^1 z_t^2(\sigma,t)d\sigma + 2\chi\frac{d}{dt}\left(\int_0^1 \sigma z_t z_\sigma d\sigma\right).\end{aligned}$$

Integration by parts + boundary conditions $\Rightarrow$

$$\begin{aligned}\int_0^1 z_t(\sigma,t)z_{\sigma\sigma}(\sigma,t)d\sigma &= z_t(\sigma,t)z_\sigma(\sigma,t)|_0^1 - \int_0^1 z_{t\sigma}(\sigma,t)z_\sigma(\sigma,t)d\sigma \\ &= z_t(1,t)(-kz_t(1,t)+rw(t)) - \int_0^1 z_{t\sigma}(\sigma,t)z_\sigma(\sigma,t)d\sigma.\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(2\int_0^1 \xi z_t z_\xi d\xi\right) &= -\int_0^1 (z_t^2 + a z_\sigma^2)d\sigma + z_t^2(1,\, t) \\ &\quad + a[-kz_t(1,\, t) + w(t)]^2 + 2d\int_0^1 \sigma z_t(\sigma,t)z_\sigma d\sigma.\end{aligned}$$

We find

$$\frac{d}{dt}V + 2\delta V - bw^2 = \int_0^1 \vartheta^T(\sigma,t)\Psi\vartheta(\sigma,t)d\sigma \leq 0$$

with

$$\vartheta^T(\sigma,t) = [z_t(1,t)\ z_\sigma(\sigma,t)\ z_t(\sigma,t), w(t)]$$

if

$$\Psi = \begin{bmatrix} \psi_1 & 0 & 0 & -a\chi kr + apr \\ * & \psi_2 & (2\delta + d)\chi\sigma & 0 \\ * & * & \psi_3 & 0 \\ * & * & * & -b + \chi ar^2 \end{bmatrix} < 0 \Leftarrow \Psi_{|\sigma=1} < 0$$

where

$$\psi_1 = -2akp + (1 + ak^2)\chi,$$
$$\psi_2 = -a\chi + 2\delta ap,$$
$$\psi_3 = -\chi + 2pd + 2\delta p.$$

A wave equation analysis

Theorem

Given $\delta > 0$, if $\exists p > 0, \chi > 0$ such that

$$\Psi_{|\sigma=1,k=k_i} < 0,\ i = 0, 1,\ \begin{bmatrix} a_1 p & \chi \\ \chi & p \end{bmatrix} > 0$$

then

$$\int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]d\sigma \leq \frac{\alpha_2}{\alpha_1} e^{-2\delta(t-t_0)} \int_0^1 [\dot{\zeta}(\sigma)^2 + \zeta_1(\sigma)^2]d\sigma + \frac{b}{\alpha_1 2\delta}\bar{w}^2$$

with

$$\alpha_1 = \lambda_{\min} \begin{bmatrix} ap & 0 \\ 0 & p \end{bmatrix},\quad \alpha_2 = \lambda_{\max} \begin{bmatrix} ap & \chi \\ \chi & p \end{bmatrix}.$$

is satisfied.

A numerical example

For the parameter values given in (Challamel, 1999) and $\beta = 0$:

$$\begin{aligned} G &= 79.3x10^9 N/m^2, \quad I = 0.095 Kg \cdot m, \\ T &= 3000 N \cdot m, \quad J = 1.19x10^{-5} m^4, \\ L &= 3145 m, \end{aligned}$$

Difference equation approach $\Rightarrow$

$$\begin{aligned} |z_\sigma(1,t)| \leq\ & 0.9979 e^{-0.2006t} [|\zeta_1(\xi/0.9979)| \\ & + 1.0021 \left|\dot{\zeta}(\xi/0.9979)\right|] + 0.0033\bar{w}. \end{aligned}$$

$$\begin{aligned} |z_t(1,t)| \leq\ & e^{-0.2006t} [|\zeta_1(\xi/0.9979)| \\ & + 1.0021 \left|\dot{\zeta}(\xi/0.9979)\right|] + 0.0033\bar{w}. \end{aligned}$$

A numerical example: Wave equation approach

The wave equation approach leads to

Case	1	2	3	4	5
δ	0.08	0.06	0.04	0.01	0.0001
b	3.2521	1.0707	1.2145	1.5221	1.7951

For $\delta = 0.04$

$$\int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]d\sigma \leq 1.1909 e^{-0.08t} \int_0^1 [\zeta_1^2(\sigma) + \dot{\zeta}^2(\sigma)]d\sigma + 11.9944\bar{w}^2.$$

The *difference equation* approach leads to an ultimate bound for the main variable of interest, the *angular velocity at the drill bottom* $z_t(1,t)$, while the *wave equation* model provides the bound on the *energy* $\int_0^1 [z_\sigma^2(\sigma,t) + z_t^2(\sigma,t)]d\sigma$.

Recent results

on *Robust Stability and Control of Distributed Parameter Systems*

- Boundary value H_∞ control of the heat/wave eqs
  (with Y. Orlov, Aut09).
  Thanks to *J.-P. RICHARD* for our visits to Ecole Centrale de Lille.
- 2-nd order evolution eqs with *boundary tvr delays*
  (with S. Nicaise & J. Valein, SICON 10)
  Thanks to *M. DAMBRINE* for my visits to Valenciennes Uni.
- *Sampled-data control* of 1-d semilinear heat equation
  under the discrete in space and in time measurements
  (with A. Blighovsky, IFAC Congress 2011)

CONCLUSIONS

Many important plants (flexible manipulators and heat transfer processes) are governed by PDEs and described by uncertain models.

- The existing results
  [Bensoussan et al 1993], [Curtain & Zwart, 1995],
  [Foias et al. 1996] , [van Keulen 1993]
  on robust control of Distributed Parameter Systems (DPS) are devoted to the linear case.
- The LMI approach (Fridman & Orlov, Aut09a,b) is appropriate for nonlinear distributed parameter models and provides the desired system performance in spite of significant uncertainties.

As it happened with Time-Delay systems, LMIs are expected to provide effective tools for robust control of Distributed Parameter Systems.