Robust stability of time-delay systems

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Outline

Stability of Time delay systems

Let consider the following time delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - \mathbf{h}), \ \forall t \geq 0 \\ x(t) = \phi(t), \ \forall t \in [-h, 0] \end{cases} \tag{1}$$

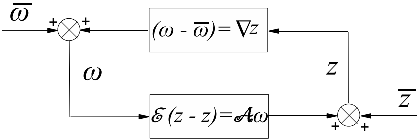
⋆ **h** is the delay and is possibly time-varying.
⋆ Goal : Give conditions on h for finding the largest interval $[h_{\min} \ h_{\max}]$ such that for all h in this interval the delay system is stable.
⋆ If $h(t)$ is time-varying, conditions will also depend on an upperbound d of $|\dot{h}(t)|$.

Previous work

Numerous tools for testing the stability of linear time delay systems have been successfully exploited :

- Direct approach using pole location [Sipahi2011].
  - ⊕ It can lead to an analytical solution...
  - ⊖ ...But it's only for **constant delay**,
  - ⊖ and robustness issues are still an open question.
- A Lyapunov-Krasovskii /Lyapunov- Razumikhin approach [Gu03, Fridman02, He07, Sun2010 ...].
  - A general L.K. functional exists but difficult to handle.
    $\Longrightarrow$ see the work of [Gu03] for a discretized scheme of the general L.K. functional.
  - Choice of more simple and then more conservative L.K. functional.
- Input - Output Approach
  - Small gain theorem [Zhang98,Gu03 ...],
  - IQC approach [Safonov02, Kao07],
  - Quadratic separation approach.
  - ⊕ It works either for constant or time varying delay systems,
  - ⊕ Robustness issue is straightforward,
  - ⊖ ...But some conservatism to handle.

Stability analysis using quadratic separation



Stability analysis of an interconnection between a linear transformation and an uncertain relation ∇ belonging to a given set $\mathbb{W}$.

- Whatever bounded perturbations $(\bar{z}, \bar{w})$, internal signals have to be bounded.
- Stability of the interconnection $\Leftrightarrow$Well-posedness pb[Safonov87].
- Separation of the graph of the implicit transformation and the inverse graph of the uncertain transformation.

$\Rightarrow$ key idea [Iwasaki98] for classical linear transformation, the well posedness is assessed losslessly by a quadratic separator (quadratic function of z and w).

$\Rightarrow$ extension to the implicit linear transformation proposed by [Peaucelle07,Ariba09].

Stability analysis using Quadratic Separator

Theorem ([Peaucelle07])

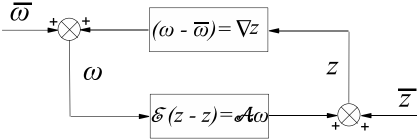
The uncertain feedback system of Figure 1 is well-posed and stable if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{2}$$

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^{*} \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \; . \tag{3}$$

Goal:Develop an interconnected system to use this theorem, i.e. artificially construct augmented systems to develop less conservative results.

Stability analysis using Integral Quadratic Separator



$\star$ ∇ are composed either of uncertainties or operators (see also [Peaucelle09]).
$\Rightarrow$ Rewriting of the main theorem using a scalar product.

Theorem ([Peaucelle09, Ariba09])

The interconnected system is stable if there exists a matrix $\Theta = \Theta'$ *s.t.*

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp'} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \qquad (4)$$

$$\forall u \in L_{2e}, \forall T > 0, \ \langle \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T, \Theta \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T \rangle \leq 0 \qquad (5)$$

with $\langle , \rangle$ the inner product of L_2.

Procedure

1. Define an appropriate modeling of time delay system by constructing the linear transformation defined by the matrices $\mathcal{E}, \mathcal{A}$, and the relation ∇, composed with chosen operators.
2. Define an appropriate separator a matrix Θ satisfying the constraint :

$$\forall u \in L_{2e}, \forall T > 0, \ \langle \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T, \Theta \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T \rangle \leq 0 \tag{6}$$

   The constraints are then verified by construction.
3. Solve the inequality :

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \tag{7}$$

   which proves the stability of the interconnection and the time delay system.

Procedure

1. Define an appropriate modelling of time delay system by constructing the linear transformation defined by the matrices $\mathcal{E}, \mathcal{A}$, and the relation ∇, composed with chosen operators.
2. Define an appropriate separator a matrix Θ satisfying the constraint :

$$\begin{bmatrix} 1 \\ \nabla \end{bmatrix}^* \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \ . \tag{8}$$

   The infinite numbers of constraints are then verified by construction.
3. Solve the inequality :

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \tag{9}$$

   which proves the stability of the interconnection and the time delay system.

The constant delay case

Use Theorem 1 to the time-delay system $\Longrightarrow$

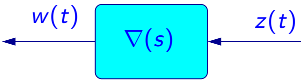
1. Rewrite system (1) as an interconnected feedback of Figure 1.
2. Embed the integrator and delay operators in an uncertain operator $\mathbb{V}$.

- ⋆ Consider the integrator s^{-1} as an uncertain operator such that $s \in C^+$, *i.e.* there is no poles in C^+.
- ⋆ Unlike IQC approaches, we do not consider dynamical system but a linear (possibly singular) transformation. The operator s^{-1} has to be considered as an uncertainty (see [Iwasaki98]).

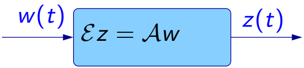
A first example (1)

⋆ We introduce $\delta_0(s) = e^{-hs}$ and $\delta_1(s) = \frac{1-e^{-hs}}{hs}$.
⋆ From the initial equation $\dot{x}(t) = Ax(t) + A_d x(t-h)$, we get

$$\overbrace{\begin{bmatrix} x(t) \\ x(t-h) \\ \frac{x(t)-x(t-h)}{h} \end{bmatrix}}^{w(t)} = \overbrace{\begin{bmatrix} s^{-1}1 & 0 & 0 \\ 0 & \delta_0(s)1 & 0 \\ 0 & 0 & \delta_1(s)1 \end{bmatrix}}^{\nabla} \overbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix}}^{z(t)}$$



$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x}(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & A_d & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & h1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x(t) \\ x(t-h) \\ \frac{x(t)-x(t-h)}{h} \end{bmatrix}$$



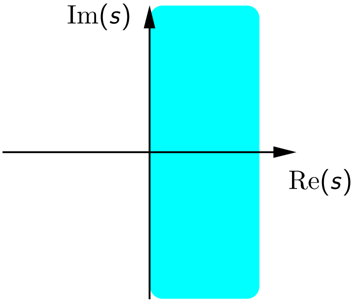
⋆ The interconnexion being established, we have to caracterise ∇ via un separator Θ.

A first example (2)

⋆ Consider then $\mathbb{W} = \mathrm{diag}\left(s^{-1}1_n,\ \delta_0(s) = e^{-hs}1_n,\ \delta_1(s) = \frac{1-e^{-hs}}{hs}1_n \right)$.

⋆ $s \in C^+$, $s^{-1} + s^{-1*} > 0$,

$$\begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix}^* \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ s^{-1}1_n \end{bmatrix} < 0.$$



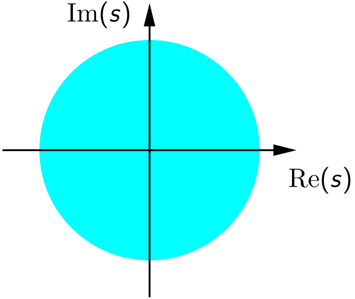
A first example (2)

⋆ Consider $\mathbb{W} = \mathrm{diag}\left(s^{-1} 1_n,\ \delta_0(s) = e^{-hs} 1_n,\ \delta_1(s) = \frac{1-e^{-hs}}{hs} 1_n \right).$

⋆ $s \in C^+$, $|\delta_0(s)| < 1$,

$$\begin{bmatrix} 1_n \\ \delta_0(s) 1_n \end{bmatrix}^* \begin{bmatrix} -Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_0(s) 1_n \end{bmatrix} < 0.$$

$\rightarrow$ δ_0 is embedded in a norm-bounded uncertainty.



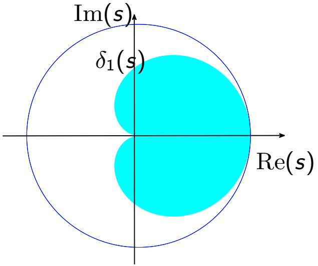
A first example (2)

⋆ Consider $\mathbb{W} = \mathrm{diag}\left(s^{-1}1_n,\ \delta_0(s) = e^{-hs}1_n,\ \delta_1(s) = \frac{1-e^{-hs}}{hs}1_n\right)$.

⋆ $s \in C^+$, $|\delta_1(s)| < 1$,

$$\begin{bmatrix} 1_n \\ \delta_1(s)1_n \end{bmatrix}^* \begin{bmatrix} -R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1_n \\ \delta_1(s)1_n \end{bmatrix} < 0.$$

$\rightarrow$ δ_1 is embedded in a norm-bounded uncertainty.



A first example (3)

⋆ Gathering all these inequalities → conservative choice of Θ.

⋆ Solving the LMI $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0$, proves the stability of the time delay system.

- The results could be interpreted with the use of a L.K. functional.

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^{t} \int_{\nu}^{t} \dot{x}(\omega)^T R \dot{x}(\omega) d\omega d\nu + \int_{t-h}^{t} x(\omega)^T Q x(\omega) d\omega$$

- The proposed technic leads to classical Lyapunov-Krasovskii functionals [Fridman02, Xu05, Suplin06, Gu03].
- It can be shown that these results are in fact equivalent to the proposed result.
- QS gives explanations of the conservatism sources for the L.K. approach.
  - Covering of the delay operators by norm bounded uncertainties.
  - Choice of conservative separators.

How to reduce conservatism ?

⋆ Let us remark firstly that

$$e^{-hs} = 1 - hs\delta_1(s).$$

⋆ δ_1 is related to the Lagrange remainder of e^{-hs} and it operates on $\dot{x}(t)$.

Idea

Consider higher order Lagrange expansion and replace the delay operator by a polynomial and the lagrange remainder.

In order to use the lagrange expansion, one needs:

- to artificially augment the system model with higher derivatives of the state.
- To have some knowledge on the remainder representative $\delta_i(s, h)$. The information to be used in the following is of norm-bounded type.

Taylor expansion of e^{-hs}

1. The Taylor series about $h_0 = 0$ of the function $h \to e^{-sh}$:

$$e^{-sh} = \sum_{i=0}^{k-1} \frac{1}{i!}(-sh)^i + R_k(s,h) \tag{10}$$

   where $R_k(s,h)$ is the Lagrange remainder.

2. Introducing new operator $\delta_i(s,h) = i!(-sh)^{-i} R_i(s,h)$ and the $r_i(t)$ signal such that:

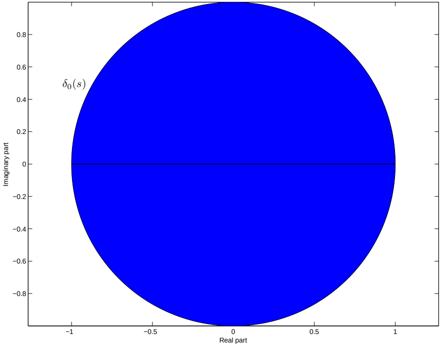
$$\delta_i(s,h)[v^{(i)}(t)] = r_i(t).$$

   then

$$\begin{aligned} v(t) &= v(t-h) + \tau r_1(t), \\ v^{(i)}(t) &= r_i(t) + \tfrac{h}{i+1} r_{i+1}(t). \end{aligned} \tag{11}$$

Graphical representation of the taylor remainder

- For $i = 0$ one has $\delta_0(s,h) = e^{-sh}$ which sweeps the whole unit circle for $s \in C^+$.
- for $i \geq 1$ the domain in which the $\delta_i(s,h)$ operators lies is reduced and not centered at zero.



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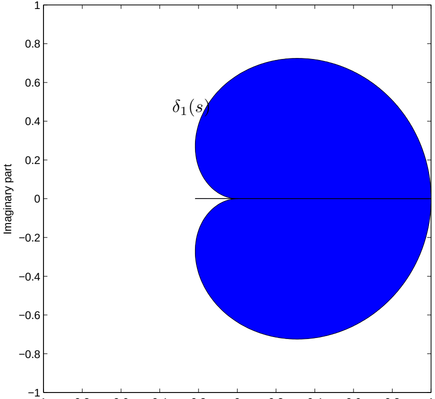

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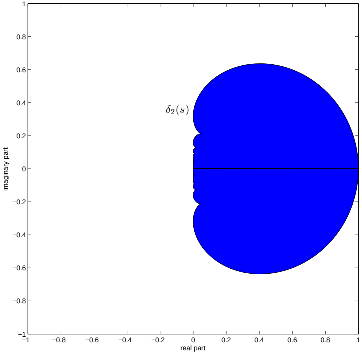

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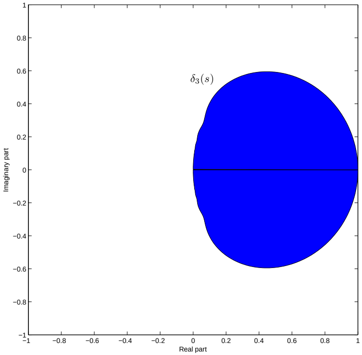


Covering set for $\delta_i(s)$

$\star$ Embed δ_k into a disk of center c_k and radius α_k:

Definition

Two sequences $\{c_k\}_{k\geq 0}$ and $\{\alpha_k\}_{k\geq 0}$ are said to be Taylor-remainder valid if $|\delta_k(s,h) - c_k| \leq \alpha_k$ for all $s \in C^+$, $\tau \geq 0$ and $k \geq 0$.

$\star$ How to find the "best" Taylor-remainder valid sequences?
$\star$ Two results are provided to construct valid sequences. The first one proves by induction that the discs may be chosen smaller as i grows.

Theorem

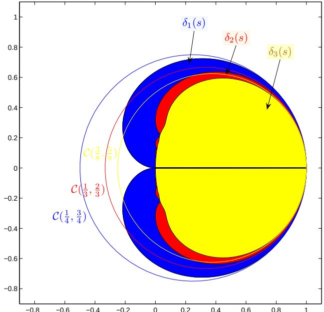
If for all $s \in C^+$ and all $\theta \in [0\ \tau]$ the complex number $\delta_i(s,\theta)$ belongs to the disc centered at c_i with radius α_i, then the same property holds for $\delta_j(s,\theta)$ with $j \geq i$.

An osculating circle for a covering set

The second result given below indicates the best disk in terms of second order approximation of the Taylor-remainder at low frequencies.

Theorem

For all $i \geq 0$, the osculating circle of $\delta_i(j\omega, h)$ at point $\omega = 0$ is centered at $c_i^{osc} = \frac{i}{2(i+1)}$ with radius $\alpha_i^{osc} = \frac{i+2}{2(i+1)}$.



- The quality of the truncated Taylor series at order k depends on the approximation of the $\delta_k(s,h)$.
- The truncation is less and less conservative as h tends to zero.

$\longrightarrow$ The methodology is applied to a fraction of the delay $\tau = h/q$.

$$v(t-h) = e^{-sh/q}[v(t-\tfrac{(q-1)h}{q})] = e^{-s2h/q}[v(t-\tfrac{(q-2)h}{q})] = \cdots = e^{-sh}[v(t)]$$

- augment the system model with all $v(t-\frac{lh}{q})$ signals were $l = \{0 \dots q\}$.

The augmented model

- Taylor series stopped at degree k and delay fractioning q.
- All possible relations described above is now defined.

1. The vectors $\tilde{x}$ and $\tilde{v}$ of all derivatives up to order $k-1$.

$$\tilde{x}(t) = \text{vec}\begin{bmatrix} x^{(k-1)}(t) & \cdots & \dot{x}(t) & x(t) \end{bmatrix}$$

$$\tilde{v}(t) = \text{vec}\begin{bmatrix} v^{(k-1)}(t) & \cdots & \dot{v}(t) & v(t) \end{bmatrix}$$

2. The vector $\hat{v}$ of delayed signals and their derivatives for all delays of the fractioning:

$$\hat{v}(t) = \text{vec}\begin{bmatrix} \tilde{v}(t - \frac{q-1}{q}h) & \cdots & \tilde{v}(t - \frac{1}{q}h) & \tilde{v}(t) \end{bmatrix}.$$

3. The vectors of remainder signals with their derivatives and the vectors of signals on which apply the operators δ_i:

$$\tilde{r}_i(t) = \text{vec}\begin{bmatrix} r_i^{(k-i)}(t) & \cdots & \dot{r}_i(t) & r_i(t) \end{bmatrix}$$

$$\tilde{v}_i(t) = \text{vec}\begin{bmatrix} v^{(k)}(t) & \cdots & v^{(i+1)}(t) & v^{(i)}(t) \end{bmatrix}.$$

The relationships between these vectors can be formulated in terms of a feedback connected system of Figure 1. Choosing the vectors:

$$\begin{aligned} z(t) &= \mathrm{vec}\left[\dot{\tilde{x}}(t) \quad \hat{v}(t) \quad \tilde{v}_1(t) \quad \cdots \quad \tilde{v}_k(t)\right] \\ w(t) &= \mathrm{vec}\left[\tilde{x}(t) \quad \hat{v}(t-\tfrac{h}{q}) \quad \tilde{r}_1(t) \quad \cdots \quad \tilde{r}_k(t)\right] \end{aligned} \tag{12}$$

the "uncertainty" that gathers all involved operators is defined as

$$\mathbb{W} = \mathrm{diag}\left[s^{-1}1_{kn} \quad \delta_0(s,\tfrac{h}{q})1_{kqp} \quad \delta_1(s,\tfrac{h}{q})1_{kp} \quad \cdots \quad \delta_k(s,\tfrac{h}{q})1_{p}\right] \tag{13}$$

The matrices $\mathcal{E}$ and $\mathcal{A}$ are constructed by the following equations:

- Augmented system equations
- Internal relationships between the signals
- Equations obtained from the Taylor series formula (11)

Theorem

Given Taylor series maximal degree k, delay fractioning q, and a Taylor-remainder valid couple (c, α), let $\mathcal{L}(k, q, c, \alpha)$ be the LMI problem composed of equation

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \quad (14)$$

with

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{12}^* & \theta_{22} \end{bmatrix}, \begin{array}{l} \theta_{11} = \mathrm{diag} \begin{bmatrix} 0 & -Q_0 & (c_1^2 - \alpha_1^2)Q_1 & \cdots & (c_k^2 - \alpha_k^2)Q_k \end{bmatrix} \\ \theta_{12} = \mathrm{diag} \begin{bmatrix} -P & 0 & -c_1 Q_1 & \cdots & -c_k Q_k \end{bmatrix} \\ \theta_{22} = \mathrm{diag} \begin{bmatrix} 0 & Q_0 & Q_1 & \cdots & Q_k \end{bmatrix} \end{array} \quad (15)$$

where the matrices P, Q_i are all symmetric, P is positive definite and the Q_i are positive semi-definite. The time-delay system (1) is stable if $\mathcal{L}(k, q, c, \alpha)$ is feasible.

Lyapunov counterpart

This approach has a LKF counterpart of the type:

$$V(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ x^{(k)}(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ x^{(k)}(t) \end{bmatrix} + \int\limits_{t-h/r}^{t} \begin{bmatrix} x(s) \\ x(s-h/r) \\ \vdots \\ x(s-\frac{r-1}{r}h)) \end{bmatrix}^T Q \begin{bmatrix} x(s) \\ x(s-h/r) \\ \vdots \\ x(s-\frac{r-1}{r}h)) \end{bmatrix}$$

$$+ \int\limits_{t-h/r}^{t} (s-h/r)^k x^{(k)\,T}(s) R x^{(k)}(s) ds$$

Theorem (IOD result)

If $\mathcal{L}(k, q, c, \alpha)$ is feasible with a Θ matrix restricted to have $Q_i = 0$ for all $i = 1 \dots k$, then the system is stable whatever the value of the delay h.

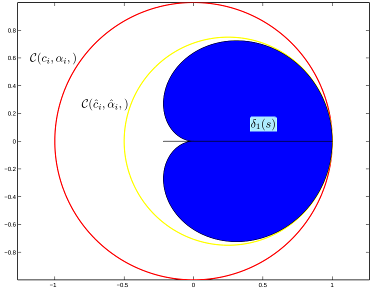
$\star$ Analogue of the result of Bliman [Bliman 01].

Theorem (DD result)

If $\mathcal{L}(k = 1, q, c, \alpha)$ is feasible for a delay $\overline{h}$, then the system is stable whatever $h \in [0\ \overline{h}]$.

Theorem

Let two couples (c, α) and $(\hat{c}, \hat{\alpha})$ such that, for all i, the disc centered at $\hat{c}_i$ with radius $\hat{\alpha}_i$ is included in the disc centered at c_i with radius α_i. In such a case, if $\mathcal{L}(k, q, c, \alpha)$ is feasible, $\mathcal{L}(k, q, \hat{c}, \hat{\alpha})$ is feasible as well.



Theorem

If $\mathcal{L}(k, q, c, \alpha)$ is feasible then for all larger degrees of the Taylor series $\hat{k} \geq k$, $\mathcal{L}(\hat{k}, q, c, \alpha)$ is feasible as well.

Theorem

If $\mathcal{L}(k, q, c, \alpha)$ is feasible then for any thinner fractioning $\hat{q} \geq q$, $\mathcal{L}(k, \hat{q}, c, \alpha)$ is feasible as well.

Time varying delay case

⋆ Adapt quadratic separation stability result.
⋆ ∇ are composed either of uncertainties or operators (see also [Peaucelle09]).
$\Rightarrow$ Rewriting of the main theorem using a scalar product.

Theorem ([Peaucelle et al 09,Ariba et al 09])

The interconnected system is stable if there exists a matrix $\Theta = \Theta'$ *s.t.*

$$\left[\begin{array}{cc} \mathcal{E} & -\mathcal{A} \end{array}\right]^{\perp'} \Theta \left[\begin{array}{cc} \mathcal{E} & -\mathcal{A} \end{array}\right]^{\perp} > 0 \qquad (16)$$

$$\forall u \in L_{2e}, \forall T > 0, \; \langle \left[\begin{array}{c} 1 \\ \mathbb{P}_T \nabla \end{array}\right] u_T, \Theta \left[\begin{array}{c} 1 \\ \mathbb{P}_T \nabla \end{array}\right] u_T \rangle \leq 0 \qquad (17)$$

with $\langle , \rangle$ the inner product of L_2.

Procedure

⋆ Time varying delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)), \ \forall t \geq 0 \\ x(t) = \phi(t), \ \forall t \in [-h, 0] \end{cases}$$

1. Define an appropriate modeling of time delay system by constructing the linear transformation defined by the matrices $\mathcal{E}, \mathcal{A}$, and the relation ∇, composed with chosen operators.
2. Define an appropriate separator a matrix Θ satisfying the constraint :

$$\forall u \in L_{2e}, \forall T > 0, \ \langle \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T, \Theta \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T \rangle \leq 0 \quad (18)$$

The constraints are then verified by construction.
3. Solve the inequality :

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0, \quad (19)$$

which proves the stability of the interconnection and the time delay system.

How to adapt the result to the time varying delay

⋆ Fractionning the delay is not so obvious. For example, applying twice $\mathcal{D}_{\frac{h(t)}{2}}$:

$x(t) \longmapsto x(t - \frac{h(t)}{2}) \longmapsto x(t - \frac{h(t)}{2} - \frac{h(t - \frac{h(t)}{2})}{2}) \neq x(t - h(t))$.

⋆ But it can be done using an extra term:

$$x(t - \frac{h(t)}{2} - \frac{h(t - \frac{h(t)}{2})}{2}) = x(t - h(t) + \delta(t)),$$

with the "fractionning error" $\delta = \frac{1}{2} \int\limits_{t-\frac{h(t)}{2}}^{t} \dot{h}(s)ds$.

This last term is then bounded and can be incorporated in a robust framework. $\rightarrow$ it is not exposed in this talk.

⋆ Augment the system using extra derivative is limited since derivative of $h(t)$ also appears.

Defining operators to model the time varying delay system

⋆ Integral operator

$$\begin{aligned} \mathcal{I} : \quad & L_{2e} \longrightarrow L_{2e}, \\ & x(t) \longrightarrow \int_0^t x(\theta)d\theta, \end{aligned} \tag{20}$$

⋆ Delay operator (or shift operator)

$$\begin{aligned} \mathcal{D} : \quad & L_{2e} \longrightarrow L_{2e}, \\ & x(t) \longrightarrow x(t-h), \end{aligned} \tag{21}$$

⋆ A Taylor remainder operator (order 1) [Peaucelle07,Kao07]:

$$\mathcal{F} = (1 - \mathcal{D}) \circ \mathcal{I}$$

⋆ A Taylor remainder operator (order 2):

$$\mathcal{H} = \mathcal{I}^2 - \mathcal{D}\mathcal{I}^2 - h(t)\mathcal{I} : \quad x(t) \longrightarrow \int_{t-h(t)}^{t} \int_{s}^{t} x(\theta)d\theta ds. \tag{22}$$

Model extension

⋆ In order to use the different operators, we need to extend the model:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), \\ \ddot{x}(t) = A\dot{x}(t) + (1-\dot{h}(t))A_d\dot{x}(t-h(t)), \end{cases} \quad (23)$$

$$E\dot{\varsigma}(t) = \bar{A}\varsigma(t) + \bar{A}_d\varsigma(t-h(t)), \quad (24)$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \\ 0 & 1 \end{bmatrix},$$

$$\bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & (1-\dot{h}(t))A_d \\ 0 & 0 \end{bmatrix}.$$

⋆ Model the augmented system (24) through the new set of operators:

$$\underbrace{\begin{bmatrix} \varsigma(t) \\ \varsigma_d(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}}_{w(t)} = \underbrace{\begin{bmatrix} \mathcal{I}1_{2n} & & & \\ & \mathcal{D}1_{2n} & & \\ & & \bar{\mathcal{F}}1_{2n} & \\ & & & \mathcal{H}1_{n} \end{bmatrix}}_{\nabla} \underbrace{\begin{bmatrix} \dot{\varsigma}(t) \\ \varsigma(t) \\ \dot{\varsigma}(t) \\ \ddot{x}(t) \end{bmatrix}}_{z(t)} \quad (25)$$

with $\varsigma_d(t) = \varsigma(t - h(t))$, $w_1(t) = \frac{\varsigma(t) - \varsigma(t-h(t))}{h(t)}$, $w_2(t) = \dot{x}(t) - \frac{x(t) - x(t-h(t))}{h(t)}$,

⋆ The feedforward equation $\mathcal{E}z(t) = \mathcal{A}w(t)$ is derived accordingly to signals $w(t)$ and $z(t)$.

Construction of a separator Θ

Conservative choice : construct a separator for each operator $\mathcal{I}, \mathcal{D}, \mathcal{F}, \mathcal{H}$ which composes ∇ and then concatenate all these relations to construct the whole separator Θ.

$\mathcal{I}$ separator An integral quadratic constraint for the operator $\mathcal{I}$ is given by $\forall x \in L_{2e}^n$ and for any positive definite matrix P,

$$\left\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{bmatrix} x_T, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{bmatrix} x_T \right\rangle \leq 0.$$

$\mathcal{D}$ separator An integral quadratic constraint for the operator $\mathcal{D}$ is given by $\forall T > 0, \forall x \in L_{2e}^n$ and for any positive matrix Q,

$$\left\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{bmatrix} x_T, \begin{bmatrix} -Q & 0 \\ 0 & Q(1-\dot{h}) \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{bmatrix} x_T \right\rangle \leq 0 \qquad (26)$$

Construction of a separator Θ, proof

$\mathcal{F}$ separator An integral quadratic constraint for the operator $\mathcal{F} = (1 - \mathcal{D}) \circ \mathcal{I}$ is given by the following inequality $\forall x \in L_{2e}^n$ and for a positive definite matrix R,

$$\left\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{bmatrix} x_T, \begin{bmatrix} -h_{max}^2 R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{bmatrix} x_T \right\rangle \leq 0,$$

where h_{max} is the upperbound on the delay $h(t)$.
$\mathcal{H}$ separator An integral quadratic constraint for the operator $\mathcal{H}$ is given by the following inequality $\forall T > 0, \forall x \in L_{2e}^n, \forall S > 0$,

$$\left\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{H} 1_n \end{bmatrix} x_T, \begin{bmatrix} -\frac{h_{max}^2}{2} S & 0 \\ 0 & 2S \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{H} 1_n \end{bmatrix} x_T \right\rangle \leq 0.$$

Stability result

⋆ A separator Θ satisfying

$$\forall u \in L_{2e}, \forall T > 0, \ \langle \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T, \Theta \begin{bmatrix} 1 \\ \mathbb{P}_T \nabla \end{bmatrix} u_T \rangle \leq 0 \tag{27}$$

has been constructed.
⇒ Using Theorem 1, original delay system is stable if

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp'} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{28}$$

⋆ It depends non linearly on h and $\dot{h}(t)$.
⋆ Using Finsler Lemma, a LMI condition, linear with respect to h and $\dot{h}(t)$, can be developped.

A constant delay example

⋆ The system of [Zhang99] is tested. The data is such that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10+K) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ K \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, A_d = BC$$

⋆ K and h are considered as uncertain parameters.

⋆ Stability test for convex set in K, h plane.

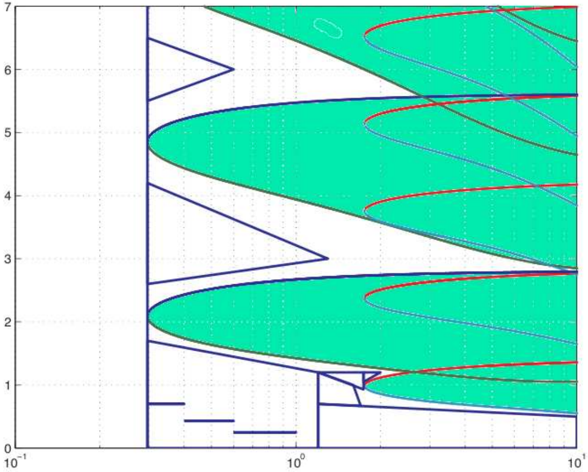


Figure: Stability regions in the (K, h) plane

⋆ Polytope of stable pockets in (K, h) plane.

A time varying delay example

Consider the following academic numerical example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \quad (29)$$

d	0	0.1	0.2	0.5	0.8	1
Fridman02	4.472	3.604	3.033	2.008	1.364	0.999
Wu04	4.472	3.604	3.033	2.008	1.364	-
Fridman06	1.632	1.632	1.632	1.632	1.632	1.632
Kao07	6.117	4.714	3.807	2.280	1.608	1.360
He07b	4.472	3.605	3.039	2.043	1.492	1.345
Ariba09	6.117	4.794	3.995	2.682	1.957	1.602
Sun10	4.476	3.611	3.047	2.072	1.590	1.529
Theorem 1	5.120	4.081	3.448	2.528	2.152	1.991

$$\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = u(t)$$

⋆ Goal: stabilizing the system using a static delayed output-feedback

$$u(t) = ky(t - h(t))$$

⋆ Choosing $k = 1$, $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - h(t))$.

- Unstable system for $h = 0$,
- analytical bounds stability $\forall h \in [0.1001, 1.7178]$ (constant delay),

	h_{min}	h_{max}
$d = 0$	0.102	1.424
$d = 0.1$	0.102	1.424
$d = 0.5$	0.104	1.421
$d = 0.8$	0.105	1.419
$d = 1$	0.105	1.418
analytical (constant case)	0.10016826	1.7178

The same example

If k is unknown, Theorem 1 allows to assess an inner (conservative) region of stability w.r.t k and $h(t)$ (for example $d = 1$).

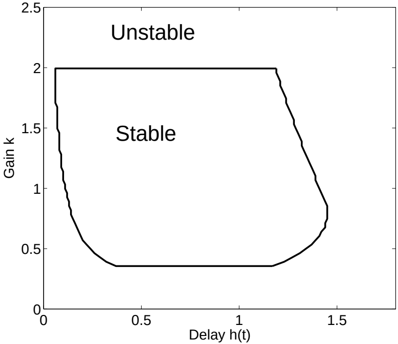


Figure: Stability region of $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = ky(t - h(t))$ w.r.t. k and $h(t)$ (for $d = 1$).

Conclusion

- Robust stability Analysis via Q.S.
- Quadratic separation framework combined with a Taylor expansion approximation of the delay operator and the fractionning of the delay.
- A sequence of LMIs have been proposed and are proved to have decreasing conservatism.