Positive trigonometric polynomials for strong stability of difference equations

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Outline

1. Neutral systems and difference equations
2. Strong stability and Hermite's condition
3. Positivity of trigonometric polynomials
4. Examples

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1. **Neutral systems and difference equations**

2. Strong stability and Hermite's condition

3. Positivity of trigonometric polynomials

4. Examples

Neutral time delay systems

A neutral system has the following form

$$\frac{d}{dt}\left(x(t)+\sum_{k=1}^{m} H_k x(t-\tau_k)\right)=A_0x(t)+\sum_{j=1}^{p} A_j x(t-\vartheta_j)$$

with time delays $\tau_k > 0, k = 1, \ldots m,\ \vartheta_j > 0, j = 1, \ldots p$, and system matrices $H_k \in \mathbb{R}^{n\times n}, k = 1, \ldots m,\ A_j \in \mathbb{R}^{n\times n}, j = 0, \ldots p$

- Lossless transmission lines (Kolmanovskii and Nosov 1986)
- Partial element equivalent circuits (PEECs) (Bellen et al. 1999)
- Combustion systems (Murray et al. 1998)
- Controlled constrained manipulators (Niculescu and Brogliato 1999)
- Boundary controlled hyperbolic PDEs (Michiels et al. 2002)
- Implementation schemes of predictive controllers (Engelborghs et al. 2001)
- Control systems with derivative feedback in vibration suppression (Vyhlídal et al. 2009)

Difference equation

The associated difference equation

$$x(t) + \sum_{k=1}^{m} H_k x(t - \tau_k) = 0$$

plays an important role when analyzing stability of the neutral time delay system

Retarded time delay systems

If $H_k = 0, k = 1, \ldots m$, the neutral system reduces to

$$\frac{d}{dt} x(t) = A_0 x(t) + \sum_{j=1}^{p} A_j x(t - \vartheta_j)$$

which is a retarded system, a more common type of time delay system

Asymptotic stability

The neutral system is stable iff all the (infinitely many) characteristic roots of the equation

$$\det\left(\lambda\left(I+\sum_{k=1}^{m} H_k e^{-\lambda\tau_k}\right)-A_0-\sum_{j=1}^{p} A_j e^{-\lambda\vartheta_j}\right)=0$$

lie in the open left half plane

Analogously, the associated difference equation is stable iff all the (infinitely many) characteristic roots of the equation

$$\det\left(I+\sum_{k=1}^{m} H_k e^{-\lambda\tau_k}\right)=0$$

lie in the open left half plane

Spectral properties

Bellman and Cooke (1963) remarked that the spectrum of a neutral system is composed of a finite number of asymptotic root chains

Some of these chains can asymptotically match exponential curves (in the same way as for retarded systems) for which $\Re(\lambda_i) \to -\infty$ as $|\lambda_i| \to \infty$

However, neutral equations may exhibit chains of characteristic roots located in vertical strips of the complex plane, i.e. $\alpha < \Re(\lambda_i) < \beta, \alpha \in \mathbb{R}, \beta \in \mathbb{R}$ as $|\lambda_i| \to \infty$

These neutral root chains arise from the fact that for large $|\lambda_i|$, in the given vertical strip, the characteristic matrix function of the difference equation in

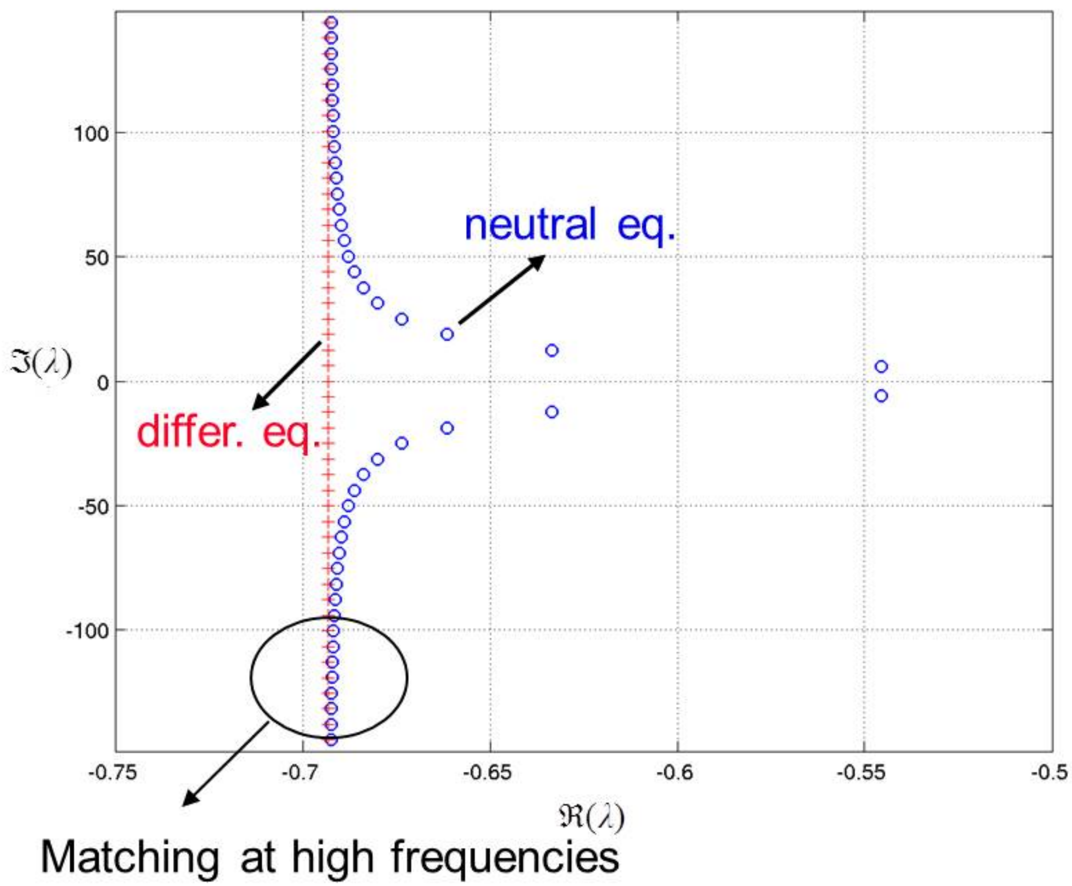
$$\det\left(\lambda\left(I+\sum_{k=1}^{m} H_k e^{-\lambda\tau_k}\right)-A_0-\sum_{j=1}^{p} A_j e^{-\lambda\vartheta_j}\right)=0$$

is of predominant order of magnitude; roots of the neutral system then tend to match roots of the difference equation

Due to this root matching, stability of the difference equation is a **necessary condition** for stability of the neutral system

Obviously, the neutral system can be unstable with infinitely many roots in the open right half plane (this can never happen for retarded systems)

Example of a typical distribution of characteristic roots



Hypersensitivity to small changes in the delays

Hale and Verduyn Lunel (1993) noticed that the distribution of the spectrum of a difference equation can be very sensitive to small changes in the delays and can cause instability

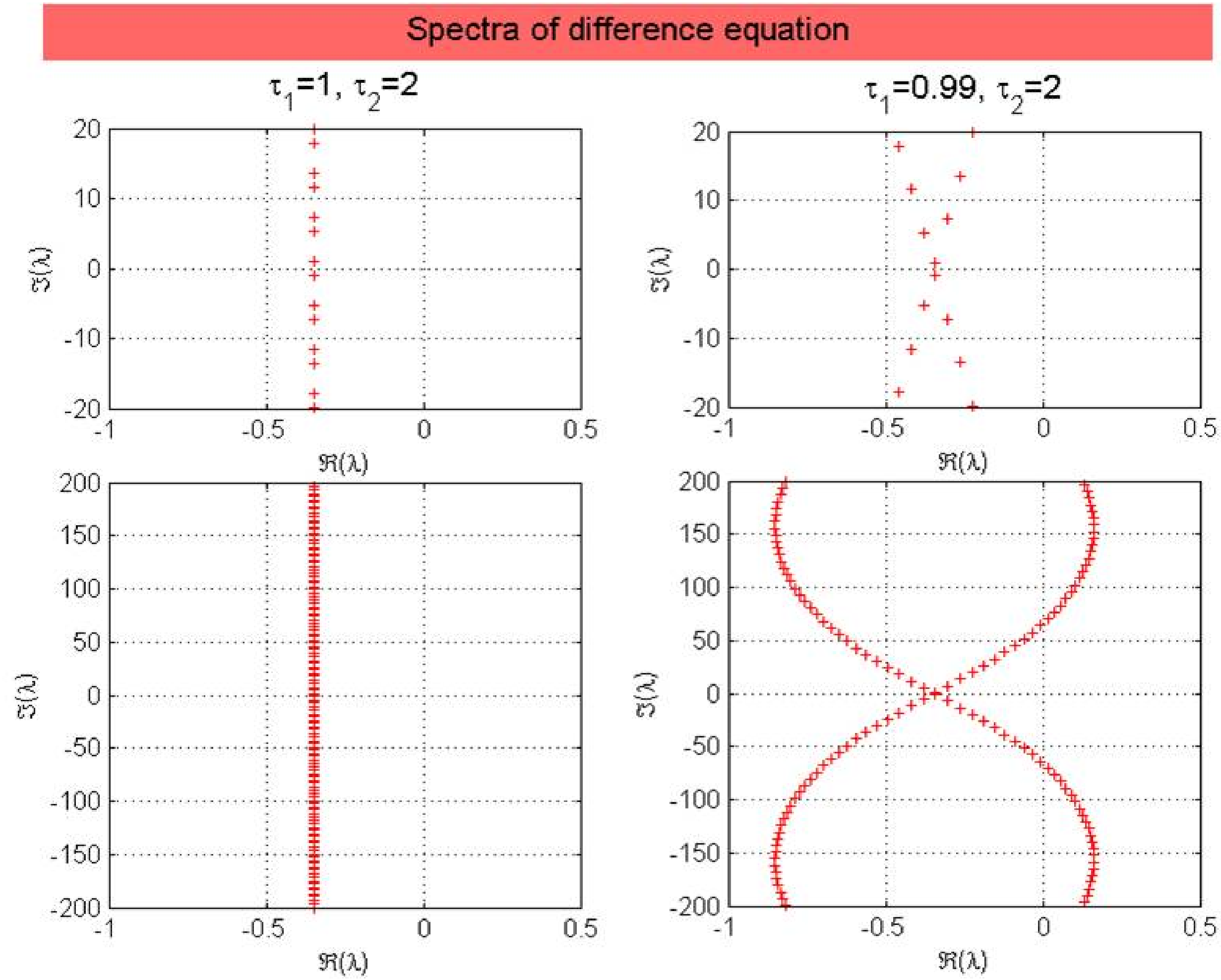
Example: consider the scalar neutral system

$$\frac{d}{dt}\left(x(t) - \frac{3}{4}x(t-\tau_1) + \frac{1}{2}x(t-\tau_2)\right) = x(t) + \frac{1}{2}x(t-\tau_2)$$

with the associated difference equation

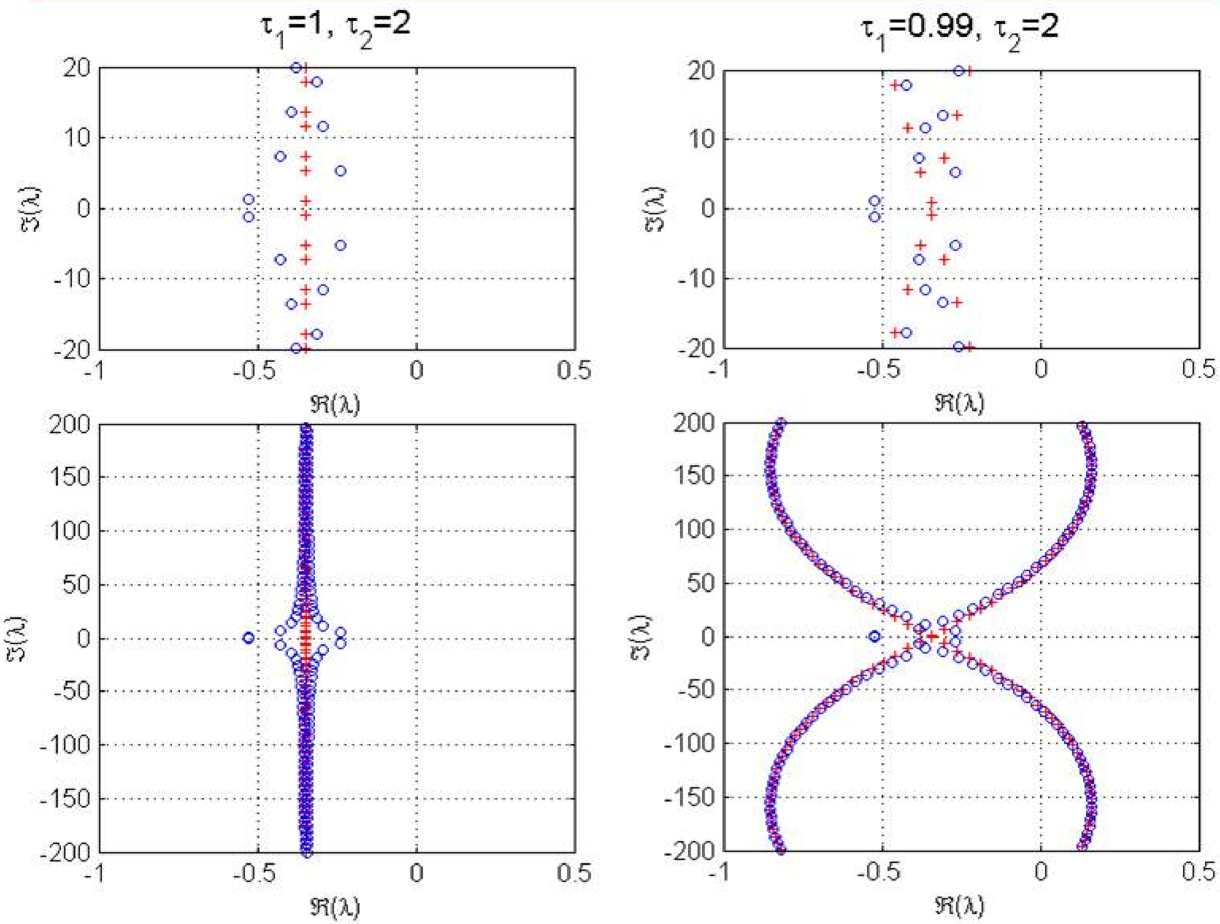
$$x(t) - \frac{3}{4}x(t-\tau_1) + \frac{1}{2}x(t-\tau_2) = 0$$

and let us analyze the spectrum of the difference equation and neutral system as the first delay changes from $\tau_1 = 1$ to $\tau_1 = 0.9$ whereas the second delay remains fixed at $\tau_2 = 2$



Stability loss due to root crossings at high frequencies

Spectra of neutral system (blue circles) and difference equation (red crosses)



The neutral system has infinitely many unstable roots

Strong stability

Although the difference equation can be stable for nominal values of the delays, stability can be lost due to **vanishingly small** changes in the delays

In order to deal with such stability hypersensitivity, the concept of **strong stability** has been introduced by Hale and Verduyn Lunel (2002):

The difference equation is strongly stable iff it is stable for the nominal delay values and also for small changes in the delays

Strong stability

The delay difference equation

$$x(t) + \sum_{k=1}^{m} H_k x(t - \tau_k) = 0$$

is **strongly stable** if and only if

$$\gamma_0 := \max_{\theta \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{k=1}^{m} H_k e^{-i\theta_k} \right) < 1$$

where r_σ denotes the spectral radius, i.e. the maximum modulus of the eigenvalues

Some well-known properties

1. Strong stability is delay independent

2. Stability of difference equation with rationally independent delays implies strong stability, and vice versa

3. In the case of one delay ($m = 1$) it holds $\gamma_0 = r_\sigma(H_1)$

4. In the case of a scalar equation ($n = 1$) it holds $\gamma_0 = \sum_{k=1}^{m} |H_k|$

5. A sufficient, but as a rule conservative, condition for strong stability is given by

$$\sum_{k=1}^{m} \|H_k\| < 1$$

where $\|.\|$ denotes the maximum singular value of a matrix

Computational issues

By homogeneity, the expression of γ_0 can be simplified to

$$\gamma_0 = \max_{\theta \in [0,\ 2\pi]^{m-1}} r_\sigma \left(\sum_{k=1}^{m-1} H_k e^{-i\theta_k} + H_m \right)$$

Finding γ_0 can be formulated as the global optimization problem of maximizing the spectral radius over $[0,\ 2\pi]^{m-1}$

As $r_\sigma(\theta)$ is in general **nonconvex** and **nonsmooth**, brute force methods have been used so far (Michiels at al. 2005-2010)

With an N-point grid for each dimension, a lower bound on γ_0 is obtained by solving N^{m-1} times an $n \times n$ eigenvalue problem

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Strong stability as robust stability

Characteristic polynomial

$$p(z) = \det(z_0 I_n + \sum_{k=1}^{m} z_k H_k)$$

homogeneous of degree n in $m+1$ variables $z_0, z_1, \ldots, z_m$

Difference equation

$$x(t) = \sum_{k=1}^{m} H_k x(t - \tau_k)$$

strongly stable iff roots of univariate polynomial

$$z_0 \mapsto p(z)$$

are in open unit disk for all $z_k = e^{j\theta_k}$, $\theta_k \in [0, 2\pi]$, $k = 1, \ldots, m$

Hermite's stability criterion

Proposed by Charles Hermite (1854) much before Routh and Hurwitz (1895)

Polynomial $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ has its roots in unit disk iff Hermite matrix $H(p) = S_1^T(p)S_1(p) - S_2^T(p)S_2(p) \succ 0$ where

$$S_1(p) = \begin{bmatrix} p_n & p_{n-1} & p_{n-2} & \\ 0 & p_n & p_{n-1} & \\ 0 & 0 & p_n & \\ & & & \ddots \end{bmatrix} \qquad S_2(p) = \begin{bmatrix} p_0 & p_1 & p_2 & \\ 0 & p_0 & p_1 & \\ 0 & 0 & p_0 & \\ & & & \ddots \end{bmatrix}$$

Explicit Lyapunov matrix depending quadratically on p

Strong stability as trigonometric polynomial matrix positivity

Apply Hermite's criterion to $z_0 \mapsto p(z)$

$$H(z_1, \ldots, z_m) \succ 0$$

for all $z_k = e^{j\theta_k}$, $\theta_k \in [0, 2\pi]$, $k = 1, \ldots, m$

In the sequel we propose to address this polynomial positivity problem with a converging hierarchy of LMI problems

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Trigonometric polynomials

Denoted

$$h(z) = \sum_{\alpha} h_\alpha z^\alpha$$

where $\alpha \in \mathbb{N}^m$ is a multi-index such that

$$z^\alpha = \prod_{k=1}^{m} z_k^{\alpha_k}$$

and $z \in \mathbb{C}^m$ is a vector of complex indeterminates such that $z \in \mathbb{T}^m$ with $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ i.e.

$$z_k = e^{j\theta_k}, \quad \theta_k \in [0,\ 2\pi]$$

Positive trigonometric polynomials

Real trigonometric polynomials $h(z) = h(z)^*$ satisfy $h_\alpha = h^*_{-\alpha}$

Positivity problem:

$$h_{\text{min}} = \min_{z \in \mathbb{T}^m} h(z)$$

Is $h_{\text{min}} > 0$?

In the sequel we propose to compute h_{min}
with converging hierarchies of:

- lower bounds via SDP
- upper bounds via EVP

Hierarchy of lower bounds via SDP

Express polynomial as a quadratic form

$$h(z) = b_k(z)^* \mathbf{X}_k b_k(z)$$

where $b_k(z)$ is a vector basis of trigonometric polynomials of degree up to k, and $\mathbf{X}_k$ is a Gram matrix

Putinar's theorem (1993): $h(z) > 0$ iff there exists a finite integer $k = d$ and a matrix $\mathbf{X}_d \succeq 0$ satisfying the above relation

Hierarchy of lower bounds via SDP

Now defining

$$\begin{array}{rcll} \underline{h}_k & = & \text{sup} & \underline{h} \\ & & \text{s.t.} & h(z) - \underline{h} = b_k^*(z)\mathbf{X}_k b_k(z) \quad \mathbf{X}_k \succeq 0 \end{array}$$

it follows that

$$\underline{h}_k \leq \underline{h}_{k+1}$$

and

$$\lim_{k \to \infty} \underline{h}_k = h_{\text{min}}$$

Computing $\underline{h}_k$ amounts to solving a semidefinite programming (SDP = LMI) problem of a special type (Toeplitz sum-of-squares)

Hierarchy of upper bounds via EVP

By definition

$$h_{\text{min}} = \min_{\mu} \int_{\mathbb{T}^m} h(z) d\mu(z)$$

where the minimum is over all probability measures on $\mathbb{T}^m$

Consider a trigonometric polynomial $q_k(z) = \mathbf{q}_k^* b_k(z)$ of degree k such that measure $\mu_k(dz) = q_k^*(dz) q_k(dz) \nu(dz)$ is absolutely continuous w.r.t. measure ν supported uniformly on $\mathbb{T}^m$

Since $\mathbb{T}^m$ is compact there is a sequence $\{\mu_k\}$ such that

$$\lim_{k \to \infty} \mu_k = \arg\min_{\mu} \int_{\mathbb{T}^m} h(z) d\mu(z)$$

Hierarchy of upper bounds via EVP

Define

$$\overline{h}_k = \min_{\mu_k} \int_{\mathbb{T}^m} h(z) d\mu_k(z)$$

where the minimum is over probability measures with polynomial densities of degree k on $\mathbb{T}^m$

Then

$$\int_{\mathbb{T}^m} h(z) d\mu_k(z) = \mathbf{q}_k^* \left(\int_{\mathbb{T}^m} h(z) b_k(z) b_k^*(z) d\nu(z) \right) \mathbf{q}_k = \mathbf{q}_k^* \mathbf{M}_k(hy) \mathbf{q}_k$$

where $\mathbf{M}_k(hy)$ is the localizing matrix (of order k of measure ν w.r.t. polynomial h) linear in moments $y_\alpha = \int_{\mathbb{T}^m} z^\alpha d\nu(z)$

Since μ_k is a probability measure

$$\int_{\mathbb{T}^m} d\mu_k(z) = \mathbf{q}_k^* \mathbf{M}_k(y) \mathbf{q}_k = 1$$

Hierarchy of upper bounds via EVP

Generalized eigenvalue problem

$$\begin{array}{rcll} \overline{h}_k & = & \min_{\mathbf{q}_k} & \mathbf{q}_k^* \mathbf{M}_k(hy)\mathbf{q}_k \\ & & \text{s.t.} & \mathbf{q}_k^* \mathbf{M}_k(y)\mathbf{q}_k = 1 \end{array}$$

with given moment matrix $\mathbf{M}_k(y) \succ 0$
and given localizing matrix $\mathbf{M}_k(hy) \succ 0$

Optimal $\mathbf{q}_k$ is a minimum eigenvector of pencil

$$z \mapsto z\mathbf{M}_k(y) - \mathbf{M}_k(hy)$$

and $\overline{h}_k$ is the minimum eigenvalue

Polynomial matrices

We want to assess whether

$$H(z) = \sum_{\alpha} z^{\alpha} H_{\alpha} \succ 0$$

so the hierarchy of SDP problems reads

$$\begin{array}{rcll} \underline{h}_k & = & \sup & \underline{h} \\ & & \text{s.t.} & H(z) - \underline{h} I = (b_k(z) \otimes I_n)^* \mathbf{X}_k (b_k(z) \otimes I_n) \quad \mathbf{X}_k \succeq 0 \end{array}$$

If $\underline{h}_k > 0$ for some k, then we are done
Otherwise we have to increase k

Similarly for the hierarchy of EVP problems

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4. **Examples**

Application example 1

Three states, two delays

$$H_1 = \begin{bmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix}$$

Applying brute force method (with $N = 360$) provides

$$\gamma_0 > 0.7507$$

in less then 0.1 seconds

However this is only a lower bound, and we cannot guarantee strong stability ($\gamma_0 > 1$)

Application example 1

Matlab script for SDP approach

```
H1=[0 0.2 -0.4;-0.5 0.3 0;0.2 0.7 0];
H2=[-0.3 -0.1 0;0 0.2 0;0.1 0 0.4];
p=sampledet({eye(3),H1,H2}); % evaluate determinant
p=p(:,abs(p(1,:))>1e-8); % remove small coefficients
H=trigoherm(p); % compute Hermite matrix
[A,b,c,K]=trigohermgram(H); % build SDP problem
[x,y,info]=sedumi(A,b,c,K); % solve SDP problem
```

M-files available for download at

homepages.laas.fr/henrion/software/trigopoly.tar.gz

Application example 1

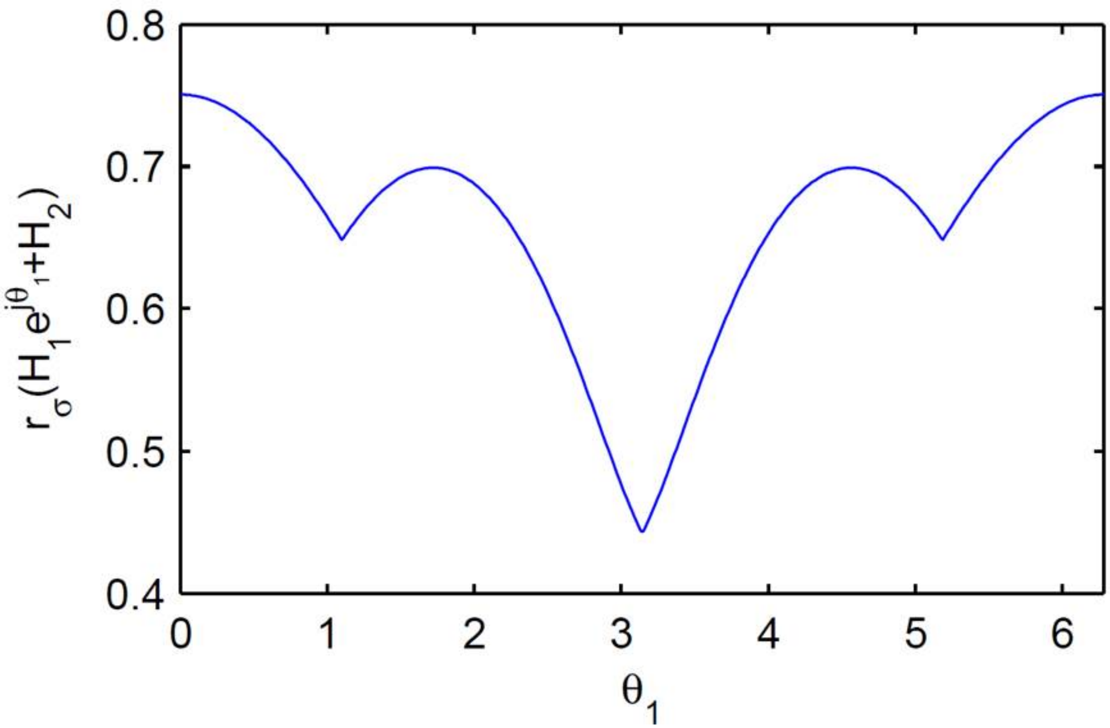
SDP problem in primal/dual form

$$\begin{array}{llcll} \min_x & c^T x & \quad & \max_y & b^T y \\ \text{s.t.} & Ax = b & & \text{s.t.} & z = c - A^T y \\ & x \in K & & & z \in K \end{array}$$

The resulting SDP problem has size `length(x)=2304`, `length(y)=225` and a positive semidefinite Gram matrix of size `K.s=48` is found after less than 0.1 seconds with SeDuMi 1.3

So we can **guarantee** that $\gamma_0 < 1$

Spectral radius $r_\sigma(\theta_1)$



Application example 2

Four states, three delays

$$H_1 = \begin{bmatrix} -0.15 & 0 & 0.32 & 0 \\ 0 & -0.07 & 0 & 0.05 \\ 0.08 & 0 & 0.04 & 0 \\ 0.2 & 0.03 & 0 & -0.13 \end{bmatrix} \quad H_2 = \begin{bmatrix} -0.02 & 0.12 & 0 & 0.25 \\ 0 & -0.05 & 0.04 & 0 \\ 0 & 0.23 & 0 & -0.3 \\ 0.19 & 0 & 0.28 & -0.09 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0 & 0 & -0.03 & 0.14 \\ 0.01 & -0.04 & 0 & 0 \\ 0 & 0 & 0.09 & 0.26 \\ 0.05 & -0.27 & -0.06 & 0 \end{bmatrix}$$

Brute force method (with $N = 360$) provides a lower bound

$$\gamma_0 > 0.6028$$

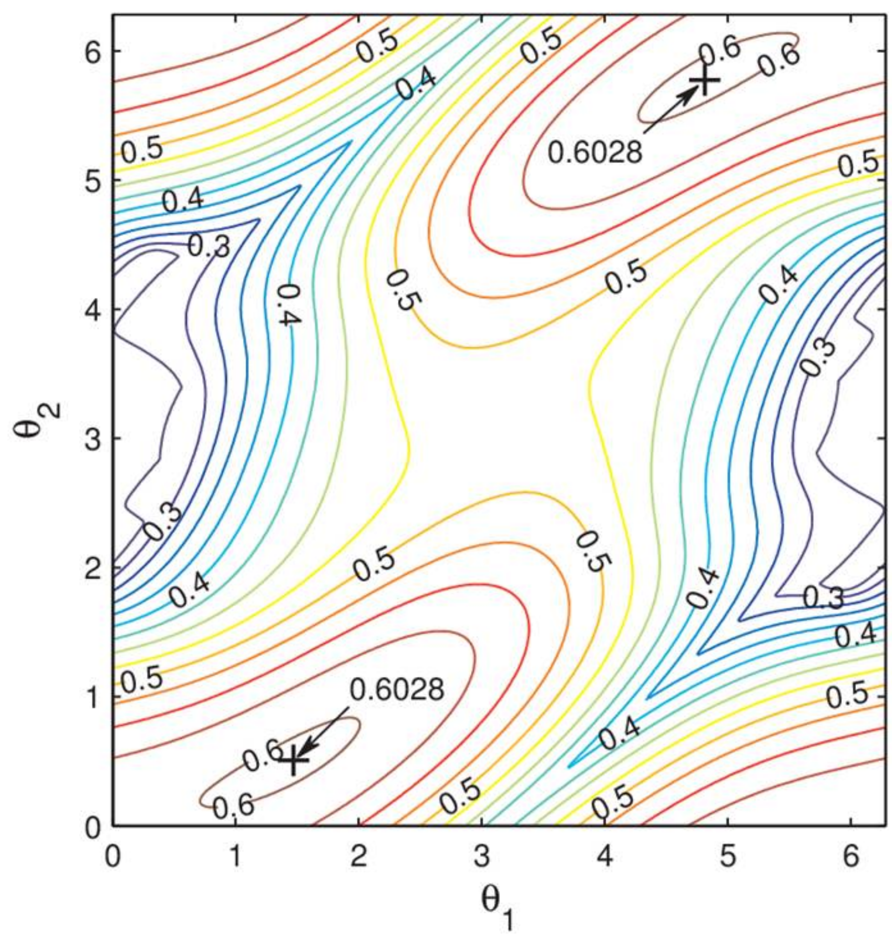
in 4.5 seconds

Application example 2

The resulting SDP problem has size `length(x)=250000`, `length(y)=5840` and a positive semidefinite Gram matrix of size `K.s=500` is found after approximately 6 minutes of CPU time

So we can **guarantee** that $\gamma_0 < 1$

Spectral radius $r_\sigma(\theta_1, \theta_2)$



Application example 3

Four states, four delays

$$H_1 = \begin{bmatrix} 0.1 & 0 & 0 & -0.2 \\ \frac{\pi}{5} & -0.1 & 0 & -0.3 \\ 0 & 0 & 0.03 & 2 \\ 0 & -e^{-1} & 0 & 0.23 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0.0456 \\ 0 & -0.33 & 0.11 & 0 \\ 0 & 1 & 0.2 & 0 \\ 0 & -e^{-3} & 0.176 & 0.73 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 0.1 & 0.65 & 0 & 0.42 \\ 0.087 & -\frac{\pi}{8} & -0.1 & 0 \\ 0 & -0.063 & 0 & 0.72 \\ 0.076 & 0.1 & 0 & -0.23 \end{bmatrix} \quad H_4 = \begin{bmatrix} -0.678 & 0 & 0 & -0.4 \\ -0.0983 & 0 & 0 & 0 \\ 0 & 0.0763 & 0 & 0.2 \\ -e^{-5} & 0 & 0.36 & 0 \end{bmatrix}$$

Applying brute force method (with $N = 360$) provides $\gamma_0 > 1.7649$ in more than 30 minutes

The resulting SDP problem is too large to be solved by standard computers