Discrete and inter-sample analysis of systems with aperiodic sampling

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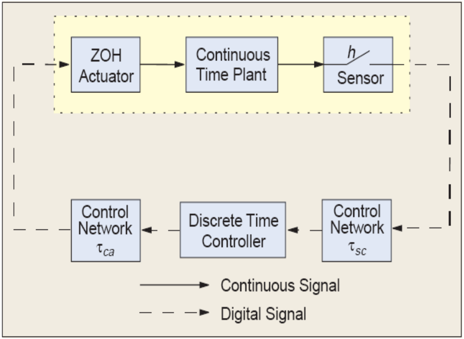
Outline

Motivating problem

Classical NCS

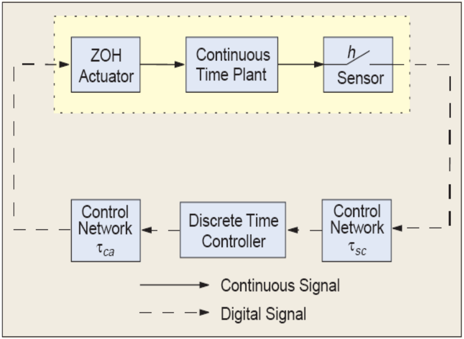


Ideal Hypothesis :

- Sampling and actuation are periodic and synchronous

Motivating problem

Classical NCS

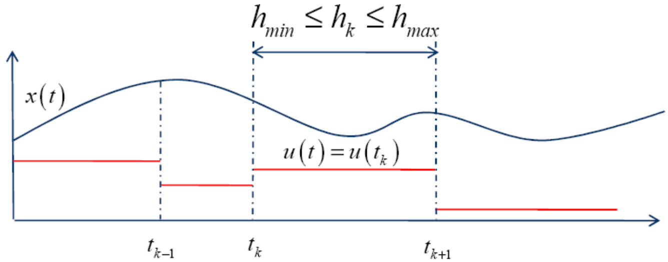


Real-time problem : the system is affected by timing problems

- sampling jitter (sensor, multitasking processors, packet dropouts in communication channels)
- unknown time varying delays

(Wittenmark, Nilsson, Torngren, 1995)

Simplified Problem Formulation



Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \ \forall t \in \mathbb{R}^+$$

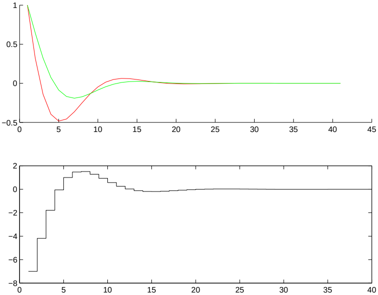
with a sampled-data control :

$$u(t) = Kx(t_k), \ \forall t \in [t_k, t_{k+1})$$

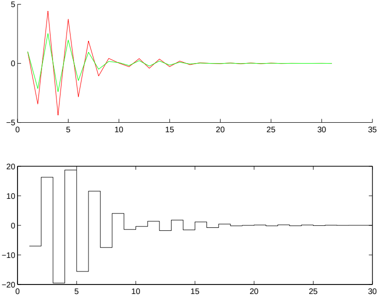
Problem : Is the system stable under sampling variations ?

Sampling jitter example (Zhang,2001)

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \;\; h_k \in \{T_1, T_2\}$$



$h = 0.18s$

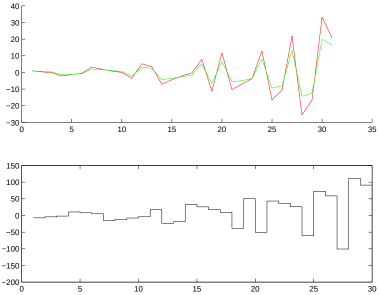


$h = 0.58s$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}; \quad K = -[1 \;\; 6]$$

Sampling jitter example (Zhang,2001) $\Rightarrow$ instability

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in \{T_1, T_2\}$$



$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}; \quad K = -[1 \;\; 6]$$

Open problem : provide tools for robust stability analysis !

Existing work

Continuous-time

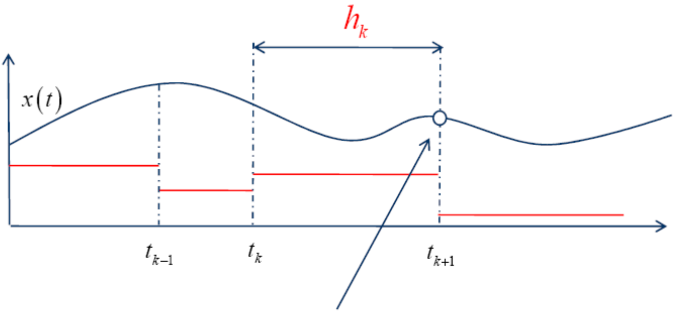
- E. Fridman et al, Automatica 2004 ; 2010 (input delay approach)
- L. Mirkin, TAC 2007 (robust control)
- J. Hespanha, SCL 2008 (impulsive delay diff. eq.)

Discrete-time

- Sala, Automatica 2004 ; Boyd, TAC 2008 (gridding approach)
- Hetel, TAC 2006, 2009 (convex embedding)
- Fujioka, TAC 2009 (gridding + robust control)

Discrete-time model

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \quad h_k \in [h_{min}, h_{max}]$$



$$x(t_{k+1}) = e^{(t_{k+1}-t_k)A}x(t_k) + \int_0^{(t_{k+1}-t_k)} e^{sA}ds\,Bu(t_{k+1})$$

$$\Longrightarrow \quad x_{k+1} = \Lambda(h_k)\,x(t_k)$$

Discrete-time model : difference inclusion

Continuous-time model

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in [h_{min}, h_{max}]$$

Equivalent discrete-time model

$$x^+ \in \mathcal{H}(x), \; \mathcal{H}(x) = \{y \; : \; y = \Lambda(h)x, \; h \in [h_{min}, h_{max}]\},$$

with

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK$$

(Continuous-time system stable iff Difference inclusion is stable)

Problems for stability analysis :

- how to deal with the uncertain matrices with exponential form ?

$$\int_0^h e^{sA} ds$$

- which class of Lyapunov functions should we chose ?

Approach

Address NCS from the point of view of polytopic difference inclusions
(Daafouz, Benoussou, SCL 2001)

Exponential uncertainty

- Integration operator

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK, \;\; h \in [h_{min}, h_{max}]$$

- For the case of quadratic Lyapunov functions $V(x) = x^T P x$:

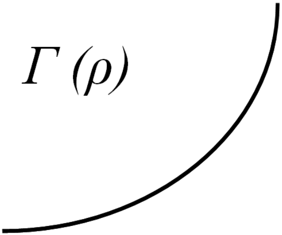
$$P > 0, \;\; \Lambda^T(h) P \Lambda(h) - P < 0, \;\; h \in [h_{min}, h_{max}]$$

Problem : infinite number of stability conditions

Exponential uncertainty

$$\Lambda(\rho) = e^{\rho A} + \int_0^\rho e^{sA} ds BK = I + \int_0^\rho e^{sA} ds \, (A + BK)$$

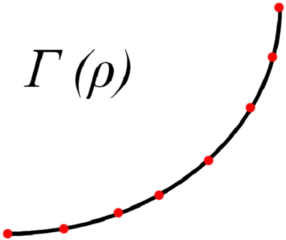
$$\Gamma(\rho) = \int_0^\rho e^{As} ds, \;\; h_{min} < \rho < h_{max}$$



curve in the space of $\mathbb{R}^{n \times n}$ matrices

Exponential uncertainty - Gridding

(Sala, Automatica, 2004)



Consider a grid on the space of parameters ρ

$$\forall \rho \in \{\rho_1, \rho_2, \ldots, \rho_N\} \subset [h_{min}, h_{max}].$$

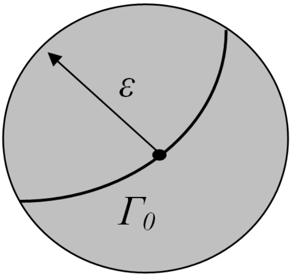
- Finite number of conditions :

$$P > 0, \;\; \Lambda^T(\rho_i) P \Lambda(\rho_i) - P < 0,$$

$$i = 1, \ldots, N$$

- Simple for illustration
- Not a sufficient condition for stability

Exponential uncertainty - Ellipsoidal embedding

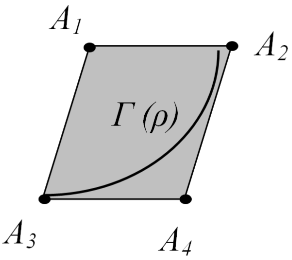


$$\Gamma(\rho) = \Gamma_0 + \Delta\Gamma$$

$$\Delta\Gamma^T \Delta\Gamma < \epsilon I$$

- LMI solution can be obtained (Gahinet, IEEE TAC, 1994), (Fujioka, IEEE TAC, 2009)
- Conservatism due to over-approximation.

Exponential uncertainty - Polytopic Embedding



$$\exists \mu_i > 0,\ \forall i = 1, \ldots, N,\ \sum_{i=1}^{N} \mu_i = 1$$

Taylor series :

- (Hetel, Daafouz, Iung, TAC 2006)

Jordan Forms :

- (Cloosterman, et. al, TAC 2009),
- (Olaru, Niculescu, IFAC World Congress 2007),

Cayley-Hamilton :

- (Gielen, et al. Automatica, 2010)

Taylor series (Hetel, Daafouz, Iung, TAC 2006)

Polynomial approximation :

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds \approx \Gamma^p(\rho) = \rho \mathbf{I} + \frac{\rho^2}{2!} A + \ldots + \frac{\rho^p}{p!} A^{p-1}$$

$$\rho \in [0, \bar{\rho}]$$

Polytopic Embedding

$$\Gamma^p(\rho) \in co \{A_1, A_2, \ldots, A_{p+1}\}$$

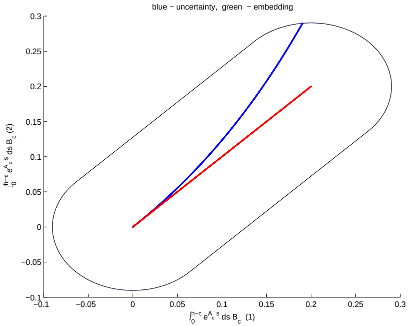
1st order approximation

Polynomial approximation :

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds \approx \Gamma^1(\rho) = \rho \mathbf{I}$$

Polytopic Embedding

$$\Gamma^1(\rho) \in co\left\{A_1, A_2\right\}, \; A_1 = \mathbf{0}, \; A_2 = \overline{\rho}\mathbf{I},$$



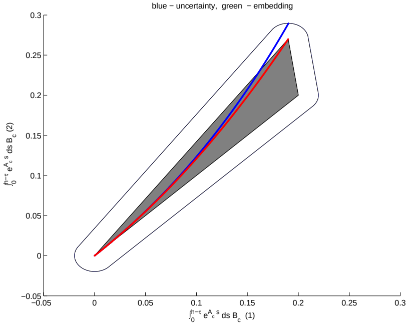
2nd order approximation

Polynomial approximation :

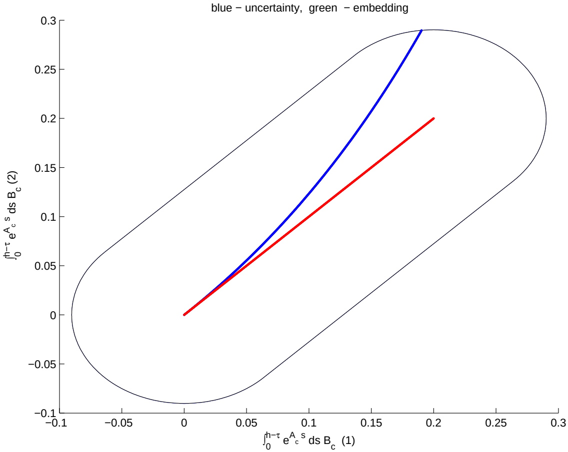
$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds \approx \Gamma^2(\rho) = \rho \mathbf{I} + \frac{\rho^2}{2!} A$$

Polytopic Embedding

$$\Gamma^1(\rho) \in co\left\{A_1, A_2, A_3\right\}, \; A_1 = \mathbf{0}, \; A_2 = \overline{\rho}\mathbf{I}, \; A_3 = \overline{\rho}^{\,2} \frac{1}{2!} A + \overline{\rho}\mathbf{I}$$

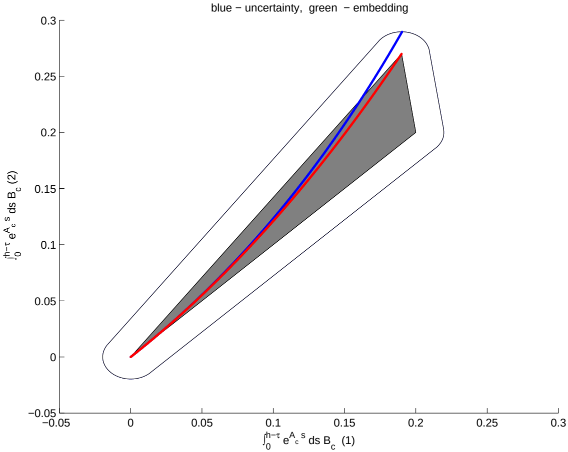


Taylor series approximation



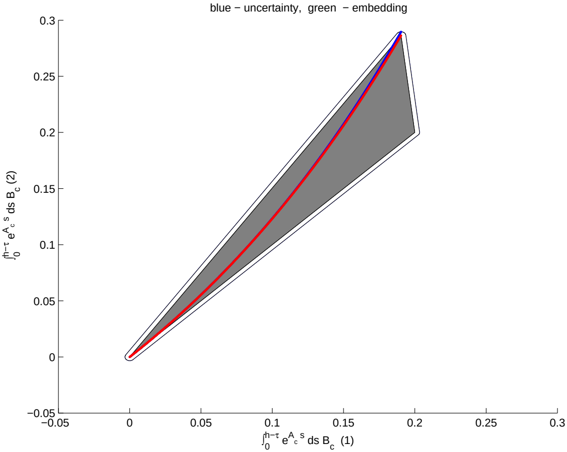
1st order approximation

Taylor series approximation



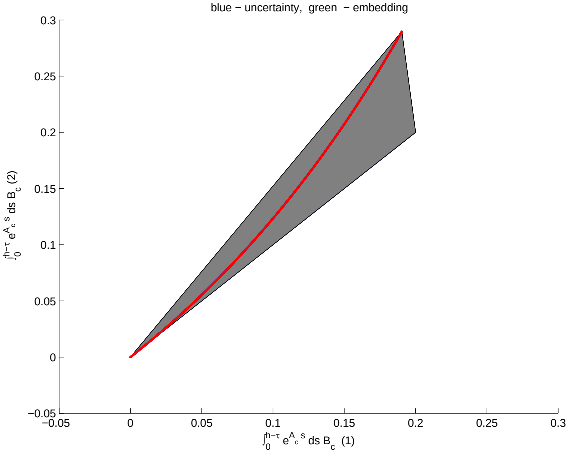
2nd order approximation

Taylor series approximation



4th order approximation

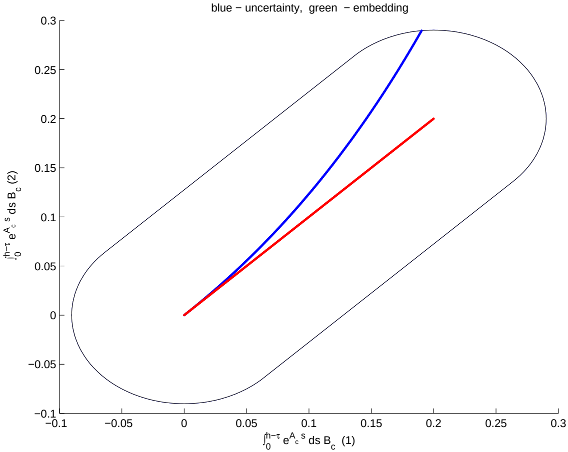
Taylor series approximation



5th order approximation

Scaling + Taylor series approximation

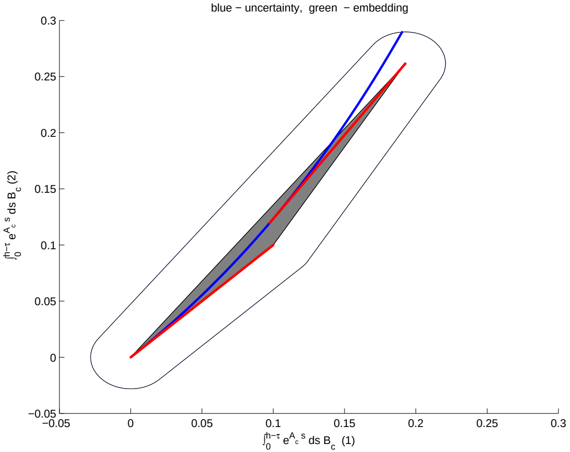
Divide the interval of analysis $[0, \bar{\rho}]$ into segments $[\rho_i, \rho_{i+1}]$



Scaling (1 segment) + 1st order approximation

Scaling + Taylor series approximation

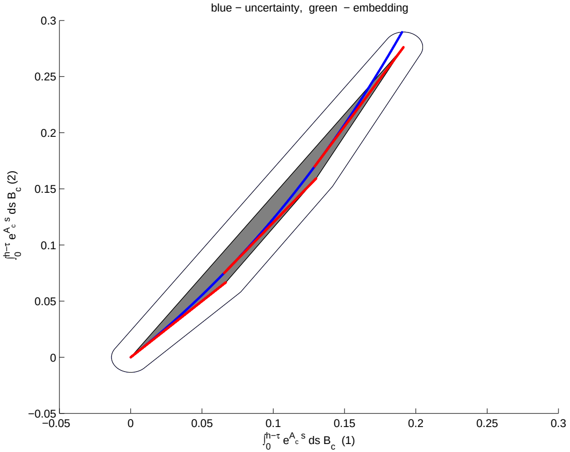
Divide the interval of analysis $[0, \bar{\rho}]$ into segments $[\rho_i, \rho_{i+1}]$



Scaling (2 segments) + 1st order approximation

Scaling + Taylor series approximation

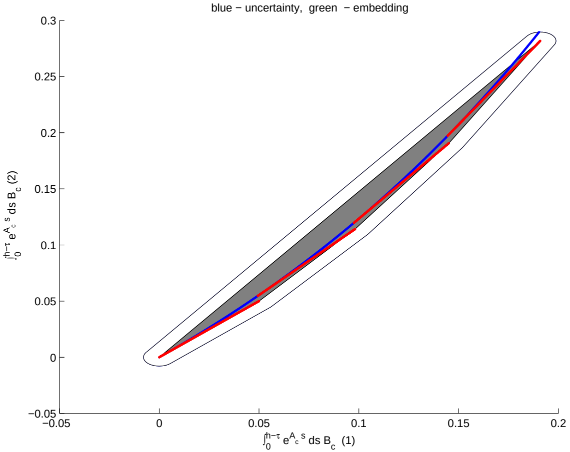
Divide the interval of analysis $[0, \bar{\rho}]$ into segments $[\rho_i, \rho_{i+1}]$



Scaling (3 segments) + 1st order approximation

Scaling + Taylor series approximation

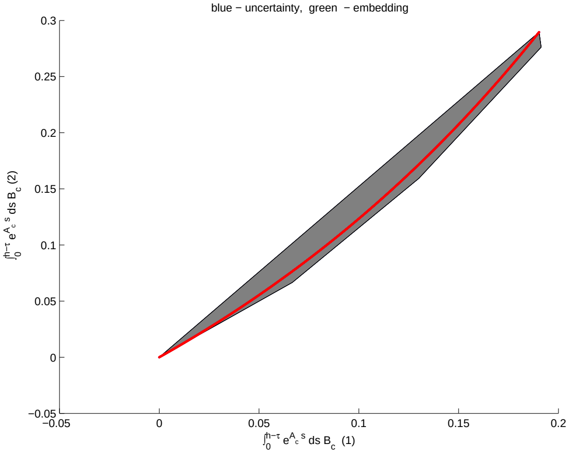
Divide the interval of analysis $[0, \bar{\rho}]$ into segments $[\rho_i, \rho_{i+1}]$



Scaling (4 segments) + 1st order approximation

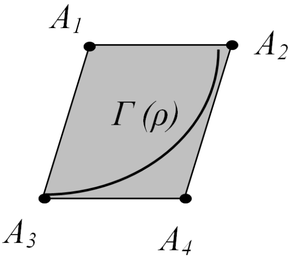
Scaling + Taylor series approximation

Divide the interval of analysis $[0, \bar{\rho}]$ into segments $[\rho_i, \rho_{i+1}]$



Scalling (3 segments) + 5th order approximation

Tractable LMI conditions



$$\Lambda^T(\rho)P\Lambda(\rho) - P < 0$$

$$\Lambda(\rho) = I + \Gamma(\rho)(A + BK)$$

- $\Lambda(\rho) \in co\{I + A_j(A + BK)\}_{j=1}^{N}$

$$P = P^T > 0$$

$$(I + A_j(A + BK))^T P (I + A_j(A + BK)) - P < 0, \ \forall j = 1, \dots, N$$

- Finite number of LMI stability conditions !

Remarks about polytopic embeddings

- The conservatism due to the use of an embedding may be tunned according to the desired numerical complexity

  (scaling factor + order of Taylor approximation)

- Polytopic + NB approach (remainder of Taylor approx.)

$$\Gamma(\rho) = \sum \mu_i \Gamma_i + \Delta\Gamma, \ \Delta\Gamma' \Delta\Gamma < \epsilon \mathbf{I}$$

  (Hetel, Daafouz, Iung, TAC 2006)

- Allow to use more efficient Lyapunov functions (Poly-quadratic, Quasi-quadratic).

Lyapunov functions

Equivalent Difference inclusion

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

with

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK$$

General Remarks on Linear Difference Inclusions (LDI) :

- Quadratic Lyapunov Functions (QLF) $V(x) = x^T P x$ are sufficient only for stability (not necessary)
- There are cases of LDI which are stable for which no QLF exists

(Dayawansa, Martin, IEEE TAC, 1999)

Lyapunov functions

Equivalent Difference inclusion

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y \ : \ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

Poly-quadratic Lyapunov functions (Daafouz, Bernoussou, SCL 2001)

$$V(x,h) = x^T P(h) x$$

Sufficient stability condition

$$\Lambda^T(h_1)P(h_2)\Lambda(h_1) - P(h_1) \prec 0, \forall h_1, h_2 \in \mathcal{T} = [h_{min}, h_{max}]$$

Lyapunov functions

With the polytopic embedding

$$\Lambda(h) \in co\mathcal{Z},\ \ \mathcal{Z} = \{Z_1, Z_2, \dots, Z_N\},$$

$$\Lambda(h) = \sum_i \mu_i(h) Z_i,\ \ \sum_i \mu_i(h) = 1,\ \ \mu_i(h) > 0$$

Poly-quadratic Lyapunov functions

$$V(x,h) = x^T \sum_i \mu_i(h) P_i x$$

Sufficient stability condition

$$Z_i^T P_j Z_i - P_i \prec 0, \forall i, j = 1, , \dots, N$$

may be used for control design (Hetel, Daafouz, Iung, IJC 2007)

Lyapunov functions

Equivalent Difference inclusion

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

General Remarks on Linear Difference Inclusions (LDI) :

- Necessary and Sufficient stability conditions : the existence of *Quasi-Quadratic* Lyapunov functions

$$V(x) = x^T \mathcal{L}_{[x]} x,$$

$$\mathcal{L}_{[x]} = \mathcal{L}_{[x]}^T = \mathcal{L}_{[ax]},\quad \forall x \neq 0, a \in \mathbb{R}, a \neq 0$$

(Molchanov and Pyatniskii, SCL 1989)
(Hetel, et al. TAC 2011)

- stability test in the literature = BMI problem

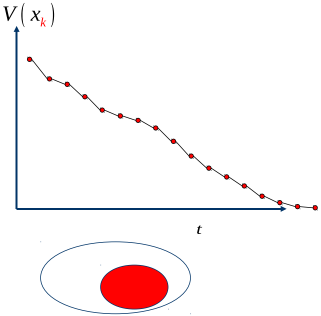
(Hu, Blanchini, Automatica, 2010)

Stability based on non-monotonous functions

(Megretzki, IEEE CDC 1994) ; (Krusezwski, Guerra, IEEE TAC 2008)

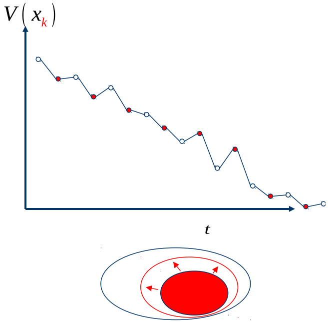
Classical approach

$$\forall x_k,\quad V(x_{k+1}) - V(x_k) < 0$$



New approach

$$\forall x_k,\quad V(x_{k+\alpha}) - V(x_k) < 0$$



Stability based on non-monotonous functions

Properties :

- $\alpha = 1$ - case of classical Lyapunov functions
- A LDI is stable iff there exist a finite $\alpha \in \mathbb{N}$ such as

$$V(x_{k+\alpha}) < V(x_k)$$

.

Stability based on non-monotonous functions

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y \ : \ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

Denote

- $\sigma = \{h^i\}_{i=0}^{\alpha-1}$ sequence of α sampling times and
- $\Phi_\sigma(\alpha)$ transition matrix associated to σ :

$$\Phi_\sigma(\alpha) = \begin{cases} \Lambda\left(h^{\alpha-1}\right)\ldots\Lambda\left(h^1\right)\Lambda\left(h^0\right), & \alpha > 0 \\ \mathbf{I}, & \alpha = 0. \end{cases}$$

with the function $V(x) = x^T P x$.
Proposition : *The equilibrium point $x = 0$ is asymptotically stable iff there exists a finite $\alpha \in \mathbb{N}^+$ s.t.*

$$\Phi_\sigma^T(\alpha) P \Phi_\sigma(\alpha) - P < 0$$

for all α lenght sequences with values in $\mathcal{T}$.

Stability based on non-monotonous functions

With the polytopic embedding

$$\Lambda(h) \in co\mathcal{Z}, \ \mathcal{Z} = \{Z_1, Z_2, \ldots, Z_N\}$$

Products of α vertices

$$\mathcal{Y}_\alpha(\mathcal{Z}) = \left\{ Y : Y = \Pi_{i=0}^{\alpha-1} Z_{\mu_i}, \ Z_{\mu_i} \in \mathcal{Z} \right\}.$$

Stability test : Given α exists $P = P^T \succ 0$ s.t.

$$P \succ Y^T P Y, \ \forall Y \ \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

- finite number of LMI (complexity to be tunned according to number of vertex N and the horizon of analysis α)

Non-monotonous functions /Quasi-Quadratic functions

Proposition. *If there exist a positive integer* α *and a matrix* $P = P^T \succ 0$ *that satisfy*

$$P \succ Y^T P Y, \ \forall Y \ \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

then there exists a composite quadratic Lyapunov function,

$$V_c(x) = x^T \mathcal{L}_{[x]} x = \max_{i=1,\dots,M} x^T L_i x, \ M = N^\alpha$$

(Hetel, et al. TAC 2011)

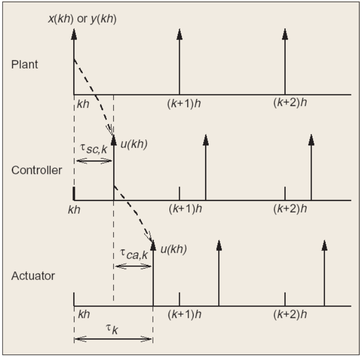
level set = intersection of ellipsoids

L_i - explicit functions of Z_i,P

Extension

- Systems with delay
- Reset systems
- Continuous-time analysis

Extension : systems with delay ($\tau_k < T$)



(Zhang,2001)

$$x_{k+1} = A_d x_k + \int_0^{h-\tau_k} e^{As} ds B u_k$$

$$+ \left(B_d - \int_0^{h-\tau_k} e^{As} ds B \right) u_{k-1}.$$

$$u_k = K x_k$$

Augmented state vector :

$$z_k = [x_k^T \; u_{k-1}^T]^T$$

$$z_{k+1} = \underbrace{\begin{bmatrix} Ad + \Gamma(h - \tau_k)BK & B_d - \Gamma(h - \tau_k)B \\ K & 0 \end{bmatrix}}_{\Lambda(\tau_k)} z_k, \; \tau_k \in [\tau_{min}, \tau_{max}]$$

LDI - similar to the case with sampling jitter

Extensions :

- Generalization for systems with large delay

$$z_k = [x_k^T \ u_{k-1}^T \ u_{k-2}^T \dots u_{k-d}^T]^T$$

(Cloosterman, Hetel, v.d.Wouw, Heemels, Daafouz, Nijmeijer, Automatica, 2010)

- Lyapunov functions $z_k^T P(\tau_k) z_k$ for LDI $\Rightarrow$ generalization of Lyapunov Krasovskii Functions

$$V(x_k, x_{k-1}, x_{k-2}, \dots, x_{k-d}) = \sum_{i=1}^{d} \sum_{j=1}^{d} x_{k-i}^T Q^{i,j}(\tau_k) x_{k-j}$$

(Hetel, Daafouz, Iung, Nonlin. Anal. Hybr. Syst., 2007 ; Hetel. et al. CDC 2009)

Extension : delay dependent controller

- Original system

$$x_{k+1} = A_d x_k + \int_0^{h-\tau_k} e^{As} ds B u_k + \left(B_d - \int_0^{h-\tau_k} e^{As} ds B \right) u_{k-1}.$$

- Equivalent open-loop system

$$z_{k+1} = \bar{A}(\tau_k) z_k + \bar{B}(\tau_k) u_k, \; z_k = [x_k^T \;\; u_{k-1}^T]^T$$

- Improved controller (adapting to delay value)

$$u(k+1) = K^x(\hat{\tau}_k) x_k + K_1^u(\hat{\tau}_k) u_k + K_2^u(\hat{\tau}_k) u_{k-1}$$

(Hetel, et al. SCL 2011)

Extension : reset systems

Sampled-data systems are particular case of reset systems :

$$\begin{array}{rcl} \dot{z}(t) & = & A_c z(t), \forall\ t \neq t_k \\ z(t_k^+) & = & A_r z(t_k) \\ t_{k+1} - t_k & \in & [h_{min}, h_{max}] \end{array}$$

Sampled-data case

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} A & B \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \ \forall\ t \neq t_k$$

$$\begin{pmatrix} x \\ u \end{pmatrix}^+ = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{K} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix},$$

Extension : reset systems

Sampled-data systems are particular case of reset systems :

$$\begin{array}{rcl} \dot{z}(t) & = & A_c z(t), \forall\ t \neq t_k \\ z(t_k^+) & = & A_r z(t_k) \\ h_k = t_{k+1} - t_k & \in & [h_{min}, h_{max}] \end{array}$$

LDI model at reset times

$$x(t_{k+1}^+) \in \mathcal{H}\left(x(t_k^+)\right),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ \ h \in [h_{min}, h_{max}]\},$$

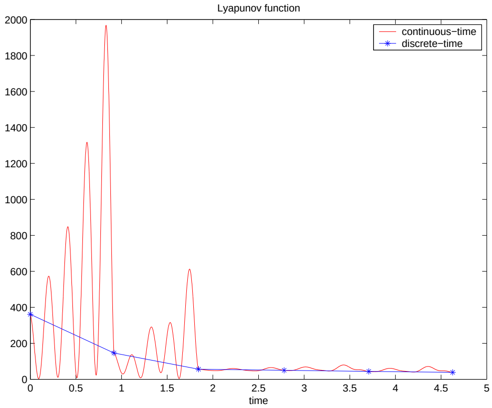
with

$$\Lambda(h_k) = A_r e^{A_c h_k}$$

(Hetel, Daafouz,Tarbouriech, Prieur, IFAC 2011)

Extension : continuous-time analysis

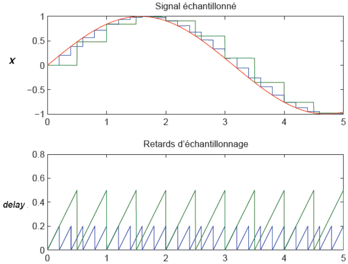
Discrete-time problem



Evolution of Lyapunov function

Extension : continuous-time analysis

Time delay model for systems with jitter



Sawtooth delay

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + BKx(t-\tau(t)), \\
\dot{\tau} &= 1, \ \forall t \in [t_k, t_{k+1}) \\
\tau(t_k^+) &= 0
\end{aligned}
$$

(Fridman, et al., Automatica, 2004)

$$x(t) = \Lambda(\tau(t))x(t-\tau(t))$$

Extension : continuous-time analysis with $V(x) = x^T P x$

$$\frac{dV(x)}{dt} = 2x^T(t)P\left(Ax(t) + BKx(t-\tau)\right) < -\alpha V(x(t)).$$

$$\begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix}^T \begin{pmatrix} A^T P + PA + \alpha P & PBK \\ K^T B^T P & \mathbf{0} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-\tau) \end{pmatrix} \prec \mathbf{0},$$

and

$$x(t) = \Lambda(\tau)\, x(t-\tau),\ \ \forall \tau \in [0, h_{max}]$$

$\Rightarrow$ LMI problem using Finsler's Lemma

Extension : continuous-time analysis with $V(x) = x^T P x$

LMI problem :

$$\exists\ P = P^T \succ 0\ G_1,\ G_2 \text{ s.t.}$$

$$\begin{pmatrix} A^T P + PA + G_1 + G_1^T + \alpha P & PBK - G_1\Lambda(\tau) + G_2^T \\ K^T B^T P - \Lambda^T(\tau) G_1^T + G_2 & -G_2\Lambda(\tau) - \Lambda^T(\tau) G_2^T \end{pmatrix} \prec \mathbf{0},$$

$\forall \tau \in [0, h_{max}]$

(Hetel et. al , TAC, 2011)

- finite number of LMI using convex polytopes
- implies that $\Lambda(\tau)$ is non-singular

Non-singularity and Quasi-Quadratic Lyapunov functions

Remark : Existence of functions of the class $V_c(x) = x^T P_{[x]} x$ is necessary when Λ is non-singular

$$\frac{dx(t)}{dt} = Ax(t) + BKx(t_k), \forall t \in [t_k, t_{k+1}),$$

$$x(t_k) = \Lambda^{-1}(\tau)x(t)$$

$$\frac{dx}{dt} \in \mathcal{H}_c(x),\ \mathcal{H}_c(x) = \{\left(A + BK\Lambda^{-1}(\tau)\right) x, \tau \in [0, h_{max}]\},$$

based on (Molchanov, Pyatniski, 1989), (Hu, Blanchini, Automatica 2010)

Extension : Quasi-Quadratic Lyapunov functions

For $V_c(x) = \max_{i=1,\dots,M} x^T L_i x$ we obtain the following set of conditions

$$\begin{pmatrix} A^T L_i + L_i A + \lambda L_i - \sum_{i \neq j} \beta_{ij} (L_j - L_i) + G_1 + G_1^T & L_i B K - G_1 \Lambda(\tau) + G_2^T \\ K^T B^T L_i - \Lambda^T(\tau) G_1^T + G_2 & -G_2 \Lambda(\tau) - \Lambda^T(\tau) G_2^T \end{pmatrix} \prec \mathbf{0},$$

$i, j = 1, \dots, M$, $\forall \tau \in [0, h_{max}]$.

- BMI conditions if L_i, β_{ij} decision variables
- LMI if L_i may be computed using discrete-time approach

Extension : Lyapunov - Razumikhin functions

- Consider the quadratic function $V(x) = x^T P x$, $P = P^T \succ 0$.
- Asymptotic stability conditions : $\dot{V}(x(t)) < 0$ whenever $V(x(t_k)) < \alpha V(x(t))$, with $\alpha > 1$
- Matrix Inequalities conditions : $P = P^T \succ 0$, a scalar $\epsilon > 0$, and matrices $G_1, G_2 \in \mathbb{R}^{n \times n}$ s.t.

$$\begin{pmatrix} A^T P + PA + \epsilon\alpha P + G_1 + G_1^T & PBK - G_1\Lambda(\theta) + G_2^T \\ K^T B^T P - \Lambda^T(\theta) G_1^T + G_2 & -G_2\Lambda(\theta) - \Lambda^T(\theta) G_2^T - \epsilon P \end{pmatrix} \prec \mathbf{0}.$$

$\forall \theta \in [0, \theta_{max}]$

Numerical examples

$$A_c = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.5 \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 1.02 & -5.62 \end{pmatrix}.$$

- $\Lambda(h)$ is Schur for any sampling interval $h \in [0, 0.46]$.
- $\Phi = (\Lambda(0.1))^6 \Lambda(0.43)$ is not Schur (exists a periodic unstable sequence).
- $h_k \in \{0.1, \; h_{max}\}$
- Exists a QLF for $h_{max} = 0.36$.
- Poly-quadratic test $h_{max} = 0.38$
- Quasi-quadratic LF with $\alpha = 7, \; h_{max} = 0.41$.

Numerical example

Consider a continuous-time system described by :

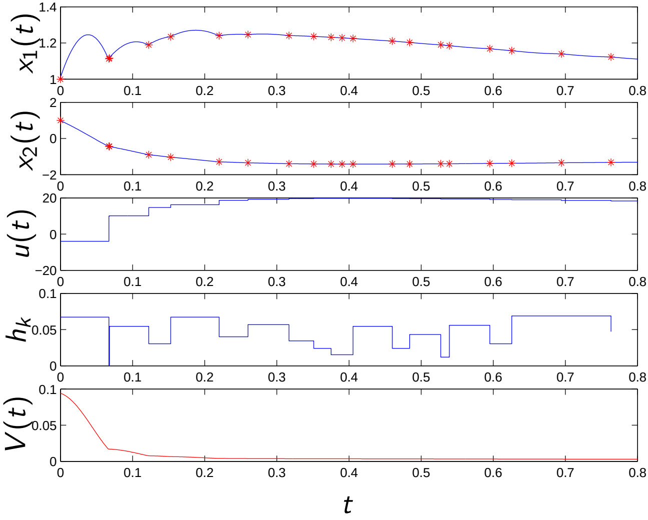
$$A = \begin{pmatrix} 1 & 15 \\ -15 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $\lambda(A) = 1 \pm 15i$
- K - obtained by pole placement : $\lambda(A + BK) = -1 \pm i$

Stability analysis comparison :

- (Naghshtabrizi, Hespanha, Teel, SCL 2008) :$h \in [0, 0.033]$
- (Fujioka, Automatica 2009) : $h \in [0, 0.07]$
- (Fridman, Automatica 2010) : $h \in [0, 0.012]$
- continuous-time + polytopic embedding : $h \in [0, 0.09]$ (singularity for 0.092)
- Lyapunov-Razumikhn + polytopic embedding : $h \in [0, 0.14]$
- discrete-time approach : $h \in [0.01, 0.15]$

Numerical example



Conclusion

- Applications of polytopic embedding methods for NCS
- Provide LMI methods for robust stability under time-varying sampling

Composite quadratic Lyapunov functions

Proposition. *If there exist a positive integer* α *and a matrix* $P = P^T \succ 0$ *that satisfy*

$$P \succ Y^T P Y, \ \forall Y \ \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

then there exists a composite quadratic Lyapunov function for the LDI,

$$V_c(x) = \max_{i=1,\ldots,M} x^T L_i x \text{ s.t. } V_c(x) > \max_{\theta \in \mathcal{T}} V_c(\Lambda(\theta)x)$$

where $L_i, \ i = 1, \ldots, M = N^{\alpha-1}$ are an enumeration of the elements in the set

$$\Omega = \left\{ Q_\sigma^Z(\alpha) = \sum_{j=1}^{\alpha-1} \left(\Pi_{r=1}^j Z_{\mu_r}\right)^T P \left(\Pi_{r=1}^j Z_{\mu_r}\right) + P, \ \sigma = \{\mu_r\}_{r=1}^{N-1} \in \{1, \ldots, N\}^\alpha \right\}.$$

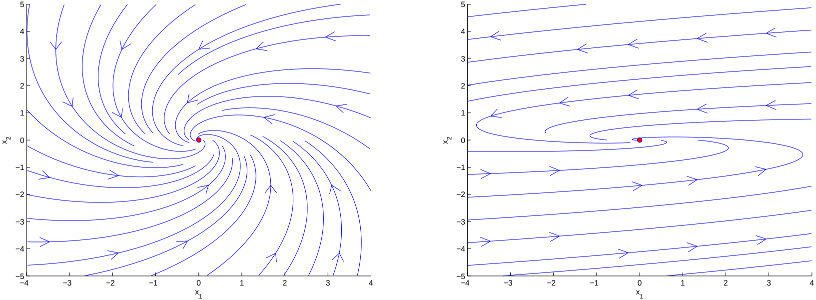
Numerical examples (Dayawansa, Martin, 1999)

Sampled-data version of

$$\dot{x} = A_\sigma x, \ \ \sigma \in \{1, 2\}$$

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \ \ A_2 = \begin{pmatrix} -1 & -8 \\ 1/8 & -1 \end{pmatrix}$$

No QLP can be found



Numerical examples (Dayawansa, Martin, 1999)

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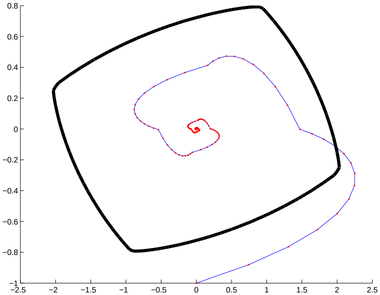


Positive Invariant Set based on Non-monotonous functions

Delay Compensation

- Discrete-time model (integration over a sampling period)

$$x_{k+1} = A_d x_k + \Omega(T - \tau_k) B u_k + (B_d - \Omega(T - \tau_k) B)\, u_{k-1}$$

$$A_d = e^{AT},\ B_d = \int_0^T e^{As} ds B,\ \Omega(\tau) := \int_0^\tau e^{As} ds.$$

- Control law

$$u_{k+1} = K_x(\hat{\tau}_k) x_k + K_u^0(\hat{\tau}_k) u_k + K_u^1(\hat{\tau}_k) u_{k-1}$$

with

$$\hat{\tau}_k = \tau_k + \delta\tau_k,\ \delta\tau_{min} \le \delta\tau_k \le \delta\tau_{max}$$

Augmented state model

Consider $\eta_k = \left[x_k' \ u_{k-1}' \ u_k'\right]'$.

$$\eta_{k+1} = \bar{A}(\tau_k)\eta_k + \bar{B}v_k$$

Delay-free LPV model

$$\bar{A}(\tau_k) = \begin{bmatrix} A_d & B_d - \Omega(T-\tau_k)B & \Omega(T-\tau_k)B \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

with

$$v_k = \mathcal{K}(\hat{\tau}_k)\eta_k$$

and

$$\mathcal{K}(\hat{\tau}_k) = \left[K_x(\hat{\tau}_k) \ K_u^1(\hat{\tau}_k) \ K_u^0(\hat{\tau}_k)\right]$$

Closed-loop model

$$\eta_{k+1} = \left(\bar{A}(\tau_k) + \bar{B}\mathcal{K}(\hat{\tau}_k)\right)\eta_k$$

(depends both on τ_k, and $\hat{\tau}_k = \tau_k + \delta\tau_k$)

$$\eta_{k+1} = \left(\bar{A}(\hat{\tau}_k) + \bar{B}\mathcal{K}(\hat{\tau}_k)\right)\eta_k + E\Omega\left(\delta\tau_k\right)\mathcal{F}(\hat{\tau}_k)\eta_k$$

where

$$E\Omega\left(\delta\tau_k\right)\mathcal{F}(\hat{\tau}_k) = \bar{A}(\tau_k) - \bar{A}(\hat{\tau}_k)$$

and

$$\Omega(\delta\tau) := \int_0^{\delta\tau} e^{As}ds.$$

Property : $\exists\ \gamma > 0\ s.t.\ \|\ \Omega(\delta\tau_k)\ \| \leq \gamma^2,\ \forall\ \delta\tau_k \in [\delta\tau_{min}, \delta\tau_{max}]$

Parametric set of LMI

Theorem

The system is stabilizable if there exist a positive scalar λ, symmetric positive definite matrices $\mathcal{S}(\cdot)$ and matrices $\mathcal{R}(\cdot), \mathcal{G}(\cdot)$, s.t. the following set of LMI is feasible :

$$\begin{bmatrix} \mathcal{G}(\theta) + \mathcal{G}'(\theta) - \mathcal{S}(\theta) & \mathcal{G}'(\theta)\bar{A}'(\theta) + \mathcal{R}'(\theta)\bar{B}' & \mathcal{G}'(\theta)\mathcal{F}'(\theta) \\ * & \mathcal{S}(\theta_+) - \lambda E E' \gamma^2 & \mathbf{0} \\ * & * & \lambda \mathbf{I} \end{bmatrix} > 0$$

for all scalars $(\theta, \theta_+) \in [0, T]^2$. The control law is given with $\mathcal{K}(\hat{\tau}_k) = \mathcal{R}(\hat{\tau}_k) \left(\mathcal{G}(\hat{\tau}_k)\right)^{-1}$.

$$V(\eta_k, \hat{\tau}_k) = \eta_k' \left(\mathcal{S}\left(\hat{\tau}_k\right)\right)^{-1} \eta_k$$

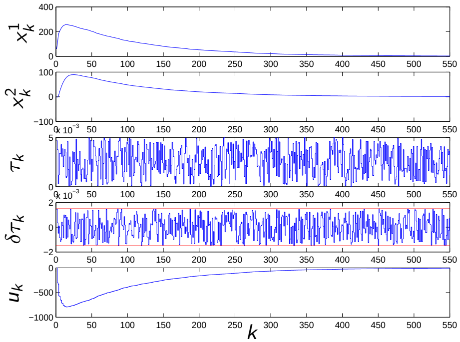
(Lyapunov function)

Example of control design for the Polytopic Approach

Motivating Example (unstable under uncertainty and delay variation)

$$A = \begin{bmatrix} 103.5 & 0 \\ 0 & -43.5 \end{bmatrix}, \quad B = \begin{bmatrix} 33.6 \\ -5.1 \end{bmatrix}, \quad T = 0.005s,$$

$$\delta\tau_k \in [-0.0015, \; 0.0015] \, .$$



Polytopic feedback with 50 vertex.

Example of control design for the Quantization Approach

- System

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

  with $a = 1$ and $b = -15$
- $[0, T]$ is divided in 3 subintervals
- $\hat{\tau}_k \in \{0.015, 0.045, 0.075\}$, $\delta\tau_{min} = -0.015$, $\delta\tau_{max} = 0.015$
- Gains

$$\begin{aligned} \mathcal{K}_1 &= [9.90 \; 9.03 \; 0.27 \; 0.68], \\ \mathcal{K}_2 &= [9.93 \; 9.03 \; 0.76 \; 0.19], \\ \mathcal{K}_3 &= [9.95 \; 9 \; 1.07 \; -0.12] \end{aligned}$$

Example of control design for the Quantization Approach

- System

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $a = 1$ and $b = -15$