

Applying Switching with Dwell Time to Robust Control of Linear Systems

L.I. Allerhand U. Shaked

Tel Aviv University, Israel

- 1 Motivation
- 2 Switched linear systems with dwell time (SLSWDL)
- 3 Robust stability of SLSWDL
- 4 L_2 -gain of SLSWDL
- 5 Parameter dependant Lyapunov functions
- 6 Robust H_∞ SF control of SLSWDL
- 7 Application to robust control
- 8 Concluding remarks

Motivation

- Robust control of linear systems with large polytopic-type uncertainties is based on solving LMIs with some common decision variables.
- The requirement for common DVs over the uncer. polytope leads to conservative results.
- The idea is to reduce this conservatism by dividing the polytope by partially overlapping regions and to assign each region to a subsystem.
- A switching is then applied according to a measurement that indicates which subsystem is active.

Motivation

- Applying the method we developed for robust stabilization and control of SLSWDL, a design method is suggested that achieves results that are better than Quadratic and/or GS designs. The stability, L_2 -gain , and S-F control of SLSWDL is considered.
- Switched systems are encountered in systems whose working conditions change very rapidly from time to time.
- A realistic case is where there is a minimum time period, the dwell time (DT), during which no switching occurs.

Motivation

- It was shown that, at least for some systems, the worst case switching law obeys some DT constraint (Margaliot and Langholz 2003).
- We shall apply switching to improve robust designs of systems without switching.
- A switching dependant Lyapunov function is used.
- We use the idea of (Boyarski and Shaked 2009).
- We specifically demand that the Lyapunov function will be non-increasing at the switching instances.
- A bounding method is then presented that

We consider the system:

$$\dot{x} = A_{\sigma(t)}x(t), \quad x(0) = x_0 \quad (1)$$

- The $\sigma(t)$ is the switching law that satisfies:
 $\tau_{h+1} - \tau_h \geq T, \quad \forall h \geq 1$ where $DT = T$, and
 the switching instants are τ_1, τ_2, \dots
- and where $A_i \in \mathcal{R}^{n \times n}, i = 1, \dots, M$ is a stability matrix which is assumed to reside in

$$\Omega_i = \left\{ A_i \mid A_i = \sum_{j=1}^N \eta^j(t) A_i^{(j)}, \sum_{j=1}^N \eta^j(t) = 1, \eta^j(t) \geq 0 \right\}$$

The best stability result achieved for **the nominal case** is of (Geromel and Colaneri 2006):

Lemma 1: Given that for some positive scalar T there exist a collection of symmetric matrices P_1, P_2, \dots, P_M of compatible dimensions that satisfy the following:

$$P_m > 0 \quad P_m A_m + A'_m P_m < 0, \quad e^{A'_m T} P_q e^{A_m T} - P_m < 0, \\ \forall m = 1, \dots, M, \quad q \neq m = 1, \dots, M$$

Then, the system is globally asymptotically stable for $DT \geq T$.

SLSWDL

For uncertain systems, a simple conservative method can be used.

Lemma 2: Assume that for some positive scalars $T, \lambda_1, \dots, \lambda_M$ there exist a collection of symmetric matrices P_1, P_2, \dots, P_M of compatible dimensions such that:

$$P_q > 0, P_q A_q^{(j)} + A_q^{(j)'} P_q + \lambda_q P_q < 0, e^{-\lambda_q T} P_q - P_m < 0, \\ \forall q = 1, \dots, M, \quad m \neq q = 1, \dots, M \quad \forall j = 1, \dots, N.$$

Then, the system is globally asymptotically stable for $DT \geq T$.

Robust stability of SLSWDL

Lemma 3: Assume that for some time interval $t \in [t_0, t_f]$, where $\delta = t_f - t_0 \exists P_1$ and $P_2 > 0$ that satisfy the following:

$$P_1, P_2 > 0, \quad \frac{P_2 - P_1}{\delta} + P_1 A + A' P_1 < 0, \\ \frac{P_2 - P_1}{\delta} + P_2 A + A' P_2 < 0.$$

Then, for the system $\dot{x} = Ax$ the Lyapunov function

$$V(t) = x'(t)P(t)x(t), \quad \text{with } P(t) = P_1 + (P_2 - P_1) \frac{t - t_0}{\delta}$$

is strictly decreasing over the time interval $[t_0, t_f]$.

The proof is by differentiating $V(t)$, and taking into account that $P(t)$ is a convex combination of P_1, P_2 over $[t_0, t_f]$.

Robust stability of SLSWDL

The extension of the latter to polytopic uncertainty is immediate, choosing the same P_1 and P_2 for all the vertices of Ω .

We next present sufficient conditions for the stability of a nominal linear switched systems.

These conditions are more conservative than those presented in (Geromel and Colaneri 2006). They are given, however, in terms of LMIs which are affine in the systems matrices, and they can thus be easily extended to the polytopic uncertainty case.

Robust stability of SLSWDL

Theorem 1: The nominal system (1) is asym. stable for any switching law with $DT \geq T > 0$ if \exists : a collection of $P_{i,k}, i = 1, \dots, M, k = 0, \dots, K$, where K is an integer, chosen a priori, s.t. $\forall i = 1, \dots, M$ the following holds.

$$P_{i,k} > 0, \quad T^{-1}K(P_{i,k+1} - P_{i,k}) + P_{i,k}A_i + A_i'P_{i,k} < 0, \\ T^{-1}K(P_{i,k+1} - P_{i,k}) + P_{i,k+1}A_i + A_i'P_{i,k+1} < 0, \quad k = 0, \dots, K-1$$

$$P_{i,K}A_i + A_i'P_{i,K} < 0, \quad P_{i,K} - P_{l,0} \geq 0, \quad \forall l \neq i.$$

Robust stability of SLSWDL

The LF would be

$$P(t) = \begin{cases} P_{i,k} + K(P_{i,k+1} - P_{i,k}) \frac{t - \tau_{h,k}}{T} & t \in [\tau_{h,k}, \tau_{h,k+1}) \\ P_{i,K} & t \in [\tau_{h,K}, \tau_{h+1,0}) \\ P_{i_0,K} & t \in [0, \tau_1) \end{cases}$$

where i is the index of the subsystem that is active at time t and $h = 1, 2, \dots$

It follows then from the first 3 ineq. that $V(t)$ is strictly decreasing during the DT. The 4th LMI guarantees that the LF is strictly decreasing for any $t \in [\tau_{h,K}, \tau_{h+1,0}]$. In the switching instants, the 5th LMI guarantees the decrease of the LF.

Robust stability of SLSWDL

The LMIs in Th. 1 are affine in the system matrices. Therefore, if the subsystems entail polytopic uncertainty, Th. 1 can provide solution to the uncertain system if the conditions hold at the vertices of all the subsystems.

In order to generalize, later, the above results to stabilization via SF the following dualization is required.

Robust stability of SLSWDL

Corollary 1: Assume that for $T > 0 \exists Q_{i,k}, i = 1, \dots, M, k = 0, \dots, K$, where K is a prechosen integer, s.t., $\forall i = 1, \dots, M$, and $j = 1, \dots, N$ the following holds:

$$Q_{i,k} > 0, \quad -\frac{K(Q_{i,k+1} - Q_{i,k})}{T} + A_i^{(j)} Q_{i,k} + Q_{i,k} A_i^{(j)'} < 0, \\ -\frac{K(Q_{i,k+1} - Q_{i,k})}{T} + A_i^{(j)} Q_{i,k+1} + Q_{i,k+1} A_i^{(j)'} < 0, \quad k = 0, \dots, K-1$$

$$A_i^{(j)} Q_{i,K} + Q_{i,K} A_i^{(j)'} < 0, \quad -Q_{i,K} + Q_{l,0} \geq 0, \\ \forall l = 1, \dots, i-1, i+1, \dots, M.$$

Then, the system (1) is globally asymptotically stable for any switching law with $DT \geq T$.

We consider the system:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{1, \sigma(t)}w(t), & x(0) &= 0, \\ z(t) &= C_{1, \sigma(t)}x(t) + D_{11, \sigma(t)}w(t) \end{aligned} \quad (2)$$

with the uncertainty polytope:

$$\tilde{\Omega}_i = \sum_{j=1}^{N_i} \eta^j(t) \tilde{\Omega}_i^{(j)}, \quad \sum_{j=1}^{N_i} \eta^j(t) = 1, \quad \eta^j(t) \geq 0$$

where:

$$\tilde{\Omega}_i^{(j)} = \begin{bmatrix} A_i^{(j)} & B_{1,i}^{(j)} \\ C_{1,i}^{(j)} & D_{11,i}^{(j)} \end{bmatrix}, \quad i = 1, \dots, M.$$

L_2 -gain

For H_∞ control we consider the following system:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{1,\sigma(t)}w(t) + B_{2,\sigma(t)}u(t), \quad x(0) = 0, \\ z(t) &= C_{1,\sigma(t)}x(t) + D_{11,\sigma(t)}w(t) + D_{12,\sigma(t)}u(t), \\ y(t) &= C_{2,\sigma(t)}x(t) + D_{21,\sigma(t)}w(t),\end{aligned}\quad (3)$$

with

$$\bar{\Omega}_i = \sum_{j=1}^{N_i} \eta^j(t) \bar{\Omega}_i^{(j)}, \quad \sum_{j=1}^{N_i} \eta^j(t) = 1, \quad \eta^j(t) \geq 0$$

where, for $i = 1, \dots, M$, $j = 1, \dots, N_i$

$$\bar{\Omega}_i = \begin{bmatrix} A_i & B_{1,i} & B_{2,i} \\ C_{1,i} & D_{11,i} & D_{12,i} \end{bmatrix}, \quad \text{and } \bar{\Omega}_i^{(j)} = \begin{bmatrix} A_i^{(j)} & B_{1,i}^{(j)} & B_{2,i}^{(j)} \\ C_{1,i}^{(j)} & D_{11,i}^{(j)} & D_{12,i}^{(j)} \end{bmatrix}$$

A_i is no longer required to be a stability matrix!

L_2 -gain

Consider the following criterion for a prescribed γ :

$$J = \int_0^{\infty} (z^T z - \gamma^2 w^T w) dt \leq 0, \quad \forall w \in \mathcal{L}_2$$

We denote

$$\bar{J} = \lim_{t \rightarrow \infty} [V(t) + \int_0^t (z^T z - \gamma^2 w^T w) ds].$$

Where $V(t) = x^T P(t)x$.

Since $V(t) \geq 0 \quad \forall t$, we have $J \leq \bar{J}$.

$V(t)$ is differentiable for $t \geq 0$, except for the switching instances, and $x(0) = 0$ thus:

$$\lim_{t \rightarrow \infty} V(t) = \sum_{h=0}^{\infty} \int_{\tau_h}^{\tau_{h+1}} \dot{V}(t) dt + \sum_{h=1}^{\infty} (V(\tau_h) - V(\tau_h^-))$$

where $\tau_0 = 0$.

L_2 -gain

If $V(t)$ is non increasing at the switching instances, we find that $V(\tau_h) - V(\tau_h^-) \leq 0 \forall h > 0$, and then:

$$\lim_{t \rightarrow \infty} V(t) \leq \sum_{h=1}^{\infty} \int_{\tau_h}^{\tau_{h+1}} \dot{V}(s) ds.$$

Denoting

$$\begin{aligned} \tilde{J} &= \sum_{h=1}^{\infty} \int_{\tau_h}^{\tau_{h+1}} \dot{V}(s) ds + \int_0^{\infty} (z^T z - \gamma^2 w^T w) ds = \\ &= \sum_{h=1}^{\infty} \int_{\tau_h}^{\tau_{h+1}} (\dot{V}(s) + z^T z - \gamma^2 w^T w) ds \end{aligned}$$

we obtain that: $J \leq \bar{J} \leq \tilde{J}$. Consequently, if $\tilde{J} \leq 0$ and the above LF does not increase at the switching instants, then $J < 0$. We thus obtain the following.

L_2 -gain

Theorem 2 ('BRL'): The L_2 -gain of the system (2) with is $< \gamma$ for $DT \geq T$ if $\exists P_{i,k}, i = 1, \dots, M, k = 0, \dots, K$ s.t. $\forall i = 1, \dots, M$, and $j = 1, \dots, N$ the following holds.

$$P_{i,k} > 0, \begin{bmatrix} \frac{K(P_{i,k+1}-P_{i,k})}{T} + P_{i,k}A_i^{(j)} + A_i^{(j)T}P_{i,k} & P_{i,k}B_{1,i}^{(j)} & C_{1,i}^{(j)T} \\ * & -\gamma^2 I & D_{11,i}^{(j)T} \\ * & * & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \frac{K(P_{i,k+1}-P_{i,k})}{T} + P_{i,k+1}A_i^{(j)} + A_i^{(j)T}P_{i,k+1} & P_{i,k+1}B_{1,i}^{(j)} & C_{1,i}^{(j)T} \\ * & -\gamma^2 I & D_{11,i}^{(j)T} \\ * & * & -I \end{bmatrix} < 0$$

$$k = 0, \dots, K - 1$$

$$\begin{bmatrix} P_{i,K}A_i^{(j)} + A_i^{(j)T}P_{i,K} & P_{i,K}B_{1,i}^{(j)} & C_{1,i}^{(j)T} \\ * & -\gamma^2 I & D_{11,i}^{(j)T} \\ * & * & -I \end{bmatrix} < 0$$

$$P_{i,K} - P_{i,0} \geq 0, \forall l = 1, \dots, i - 1, i + 1, \dots, M.$$

Parameter dependant LF

Corollary 2: The L_2 -gain of the system (2) is $< \gamma$ for $DT \geq T$ if $\exists R_i, H_i, Q_{i,k}^{(j)} = Q_{i,k}^{(j)T}, i = 1, \dots, M, k = 0, \dots, K$ s.t. $\forall i = 1, \dots, M$, and $j = 1, \dots, N$ the following holds.

$$Q_{i,k}^{(j)} > 0,$$
$$\begin{bmatrix} \frac{K(Q_{i,k}^j - Q_{i,k+1}^j)}{T} + \Psi_i^{(j)} & \bar{Q}_{i,k}^{(j)} + \bar{\Psi}_i^{(j)} & R_i^T C_{1,i}^{(j)T} & B_{1,i}^{(j)} \\ * & -H_i - H_i^T & H_i^T C_{1,i}^{(j)T} & 0 \\ * & * & -\gamma^2 I & D_{11,i}^{(j)} \\ * & * & * & -I \end{bmatrix} < 0$$

for $\bar{Q}_{i,k}^j = Q_{i,k}^j$ and $\bar{Q}_{i,k}^j = Q_{i,k+1}^j \quad k = 0, \dots, K - 1$

Parameter dependant Lyapunov

$$\begin{bmatrix} \Psi_i^{(j)} & Q_{i,K}^{(j)} + \bar{\Psi}_i^{(j)} & R_i^T C_{1,i}^{(j)T} & B_{1,i}^{(j)} \\ * & -H_i - H_i^T & H_i^T C_{1,i}^{(j)T} & 0 \\ * & * & -\gamma^2 I & D_{11,i}^{(j)} \\ * & * & * & -I \end{bmatrix} < 0$$

$$Q_{i,K}^j - Q_{l,0}^j \leq 0, \quad \forall l = 1, \dots, i-1, i+1, \dots, M.$$

where: $\Psi_i^{(j)} = R_i^T A_i^{(j)T} + A_i^{(j)} R_i + B_{2,i}^{(j)} Y_i + Y_i^T B_{2,i}^{(j)T}$,
 $\bar{\Psi}_i^{(j)} = -R_i^T + A_i^{(j)T} H_i$.

SF control

Theorem 3: The L_2 -gain of the system (3) is $< \gamma$ for $DT \geq T$ if $\exists Y_i, R_i, Q_{i,k}^{(j)} = Q_{i,k}^{(j)T}, i = 1, \dots, M, k = 0, \dots, K$, and a scalar ρ s.t. $\forall i = 1, \dots, M$, and $j = 1, \dots, N$ the following holds.

$$Q_{i,k}^{(j)} > 0,$$

$$\begin{bmatrix} \frac{K(Q_{i,k}^j - Q_{i,k+1}^j)}{T} + \bar{E}_i^{(j)} & \bar{Q}_{i,k}^{(j)} + \bar{E}_i^{(j)} & R_i^T C_{1,i}^{(j)T} + Y_i^T D_{12,i}^{(j)T} & B_{1,i}^{(j)} \\ * & -\rho R_i - \rho R_i^T & \rho R_i^T C_{1,i}^{(j)T} + \rho Y_i^T D_{12,i}^{(j)T} & 0 \\ * & * & -\gamma^2 I & D_{11,i}^{(j)} \\ * & * & * & -I \end{bmatrix} < 0$$

for $\bar{Q}_{i,k}^j = Q_{i,k}^j$ and $\bar{Q}_{i,k}^j = Q_{i,k+1}^j, k = 0, \dots, K-1$

SF control

$$\begin{bmatrix} \Xi_i^{(j)} & Q_{i,K}^{(j)} + \bar{\Xi}_i^{(j)} & R_i^T C_{1,i}^{(j)T} + Y_i^T D_{12,i}^{(j)T} & B_{1,i}^{(j)} \\ * & -\rho R_i - \rho R_i^T & \rho R_i^T C_{1,i}^{(j)T} + \rho Y_i^T D_{12,i}^{(j)T} & 0 \\ * & * & -\gamma^2 I & D_{11,i}^{(j)} \\ * & * & * & -I \end{bmatrix} < 0$$

$$Q_{i,K}^j - Q_{l,0}^j \leq 0, \quad \forall l = 1, \dots, i-1, i+1, \dots, M.$$

where $\Xi_i^{(j)} = R_i^T A_i^{(j)T} + A_i^{(j)} R_i + B_{2,i}^{(j)} Y_i + Y_i^T B_{2,i}^{(j)T}$,
 $\bar{\Xi}_i^{(j)} = -R_i^T + \rho A_i^{(j)T} R_i + \rho B_{2,i}^{(j)} Y_i$ If a solution to the latter inequalities exist, the state-feedback gain is given by $G_\sigma(t) = Y_\sigma R_\sigma^{-1}$.

Application to robust control

Example (SF control):

We consider the problem of stabilizing and attenuating disturbances acting on the longitudinal short period mode of the **F4E fighter aircraft** with additional canards (Petersen 1985).

The state space model is:

$$\frac{d}{dt} \begin{bmatrix} N_z \\ q \\ \delta_e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -30 \end{bmatrix} \begin{bmatrix} N_z \\ q \\ \delta_e \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 30 \end{bmatrix} u + Iw$$

$$z = [N_z \quad q \quad u]^T \text{ and } y = [N_z \quad q]^T.$$

In this model, N_z is the *normal acceleration*, q is the *pitch rate* and δ_e is the *elevator angle*.

Application to robust control

The parameters of the model for 4 OPs are:

O.P.	Mach	Altitude (ft)	a_{11}	a_{12}	a_{13}
1	.5	5000	-.9896	17.41	96.15
2	.9	35000	-.6607	18.11	84.34
3	0.85	5000	-1.702	50.72	263.5
4	1.5	35000	-.5162	29.96	178.9

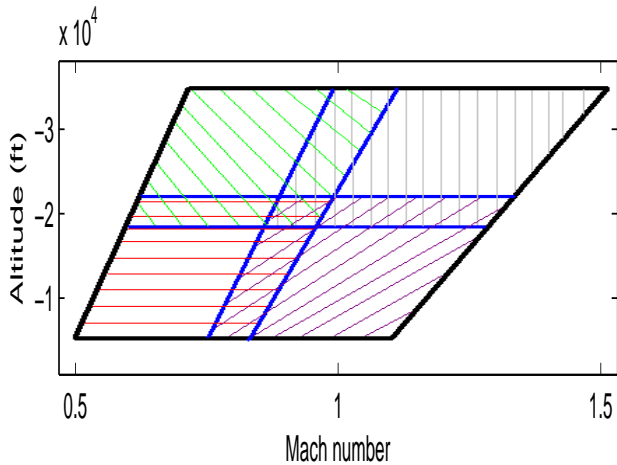
O.P.	Mach	Altit.	a_{21}	$-a_{22}$	$-a_{23}$	$-b_1$
1	.5	5000	.2648	.8512	11.39	97.78
2	.9	35000	.08201	.6587	10.81	272.2
3	0.85	5000	.2201	1.418	31.99	85.09
4	1.5	35000	-.6896	1.225	30.38	175.6

Application to robust control

It is assumed that between the OPs the parameters are a convex combination of the 4 sets of the table. 4 design methods are presented: The 1st is a robust controller with **constant gain and quadratic LF** (Petersen 1985).

The 2nd is a **GS controller with a quad. LF** (Apkarian and Gahinet 1995), and the 3rd and 4th are **switched controllers**, deigned using Th. 3.

We design the switched controller by splitting the parameters' polytope into 4 regions as described in Fig. 1.



Application to robust control

The above regions overlap, which allows some DT between switching instances.

We use Th. 3 to design a switched SF controller assuming $\sigma(t)$ is measured online, and $DT=3$ secs.

The results are:

Table: Values of γ_{min} for the F4E aircraft

Method	Quad. Robust	GS	K=1	K=10
γ_{min}	3.85	3.85	const. 2.41	const. 2.22
γ_{min}	PDL 2.91	PDL 2.90	linear 2.26	linear 2.01

The improvement is due to the decreased size of the polytopes.

Concluding Remarks

- A new method for L_2 -gain analysis of a switched system using a switching dependent LF is introduced which enables treatment of polytopic type uncertainties.
- This method is applied to the design of H_∞ SF control of systems with large uncertainties.
- The SF control is based on constant SF gains (1 for each subsystem). **Time-varying** SF gains can be obtained by letting Y_i depend on k .
- T was equally divided. Nonequal division may improve the results for the price of using BMIs.

Concluding Remarks

- The method can be applied to the solution of robust H_∞ estimation problem where a filter of general structure is sought.