Dynamics and control of (bio-)robotic locomotion: Nonlinear, nonholonomic and hybrid mechanical systems

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Background and motivation

Locomotion: ability of creature/robot/vehicle to propel itself
= internal motion/actuation + physical interaction with environment

Research Goal: analyze simple models of locomotion
as dynamics and control systems

Theoretical challenges:
mechanical modeling, nonlinear dynamics, (non-)holonomic constraints, underactuation, geometric mechanics and symmetries, hybrid dynamics (contact transitions, impacts), orbital stability, optimization, control authority & sensing, control design
Outline

1. The Twistcar toy (Ofir Chakon, MSc work)

2. Microswimmers (Emiliya Gutman, PhD work)

3. Others: slippage effects on legged locomotion
   senior project - reverse a truck & trailer
Roller Racer

Twistcar - introduction

Planar two-link model

Previous works – “Roller Racer”

$\phi$ oscillates about $\pi$

Assumptions made:

kinematic control of $\phi(t)$

**steering link** has zero mass and nonzero inertia (???)

**body** = **point mass** at back axle (tipover!?)

Used geometric mechanics, studied periodic gaits
Our research goal:
Study both angle and torque input (harmonic)
For both Twistcar and Roller-Racer configurations

Analyze influence of parameters on dynamic behavior
Assume steering link w/ zero mass+inertia, point mass body
Methods: asymptotic analysis, numerics, simple experiments
Dynamic formulation

Coordinates: \( \mathbf{q} = (x, y, \theta, \phi) \), input: \( \tau(t) \) or \( \phi(t) \)

No-skid of wheels \( \implies \) two nonholonomic constraints

\[
\mathbf{W}(\mathbf{q}) \dot{\mathbf{q}} = 0, \quad \text{where} \quad \mathbf{W}(\mathbf{q}) = \begin{pmatrix}
\sin(\theta) & -\cos(\theta) & l_1 & 0 \\
\sin(\theta + \phi) & -\cos(\theta + \phi) & l_3 - l_2 \cos(\phi) & l_3
\end{pmatrix}
\]
Dynamic formulation

Coordinates: \( q=(x,y,\theta,\phi) \), input: \( \tau(t) \) or \( \phi(t) \)

No-skid of wheels \( \rightarrow \) two nonholonomic constraints

\[
W(q)\dot{q} = 0, \quad \text{where} \quad W(q) = \begin{pmatrix} \sin(\theta) & -\cos(\theta) & l_1 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & l_3-l_2\cos(\phi) & l_3 \end{pmatrix}
\]

Constrained dynamic equations:

\[
M(q)\ddot{q} + B(q,\dot{q}) = E\tau + W(q)^T\lambda
\]

where \( M(q) = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \), \( B(q,\dot{q}) = 0 \), \( E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \), \( \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \) constraint forces
Dynamic formulation

Coordinates: \( q=(x,y,\theta,\phi) \), input: \( \tau(t) \) or \( \phi(t) \)

No-skid of wheels \( \implies \) two nonholonomic constraints

\[
 W(q) \dot{q} = 0 , \quad \text{where} \quad w(q) = \begin{pmatrix} \sin(\theta) & -\cos(\theta) & l_1 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & l_3 - l_2 \cos(\phi) & l_3 \end{pmatrix}
\]

Constrained dynamic equations:

\[
 M(q) \ddot{q} + B(q, \dot{q}) = E \tau + W(q)^T \lambda
\]

State-space formulation

\[
 \dot{x}(t) = f(x, \tau) \quad \text{where} \quad x = (q, \dot{q})^T
\]

Differentiate the constraints:

\[
 \dot{W}q + W\ddot{q} = 0
\]

\[
 \begin{pmatrix} M & -W^T \\ W & 0 \end{pmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} E \tau \\ -\dot{W}q \end{pmatrix} \quad \text{zero mass + inertia } \implies M(q) \text{ is singular (avoided in previous works) but } A(q) \text{ is not!}
\]

Angle input: \( \phi(t), \dot{\phi}(t), \ddot{\phi}(t) \) are known, \( \tau(t) \) is eliminated
Simulation results 1 (movie)

1. Twistcar, steering angle input $\phi(t) = \phi_o \sin(\omega t)$

Forward motion, diverging oscillations of body orientation
Simulation results 2 (movie)

2. Twistcar, torque input $\tau(t) = \tau_0 \sin(\omega t)$

Forward motion, decaying oscillations of body orientation + steering angle $\implies$ straight line motion
Simulation results 3 (movie)

3. Roller-Racer, angle input $\phi(t) = \pi + \phi_0 \sin(\omega t)$

Backward motion, diverging oscillations of body orientation
Simulation results 4 (movie)

4. Roller-Racer, angle input, different c.o.m. + lengths ratio

Reversed direction of motion,
diverging oscillations of body orientation
Simulation results 5 (movie)

5. Roller-Racer $\phi(0) = \pi$, torque input $\tau(t) = \tau_0 \sin(\omega t)$

Steering angle converges to oscillations about $\phi=0$  
$\implies$ back to Twistcar configuration
Twistcar - Analysis

Reduced formulation:

1. Normalize time by $1/\omega$, length by $l_1+l_2$, mass by $m$

   two nondimensional parameters: $\alpha = \frac{l_3}{l_1+l_2}$, $\beta = \frac{l_1}{l_1+l_2}$

2. Body velocities $\mathbf{v} = (v_x, v_y, \dot{\theta}, \dot{\phi})^T$, constraints $\mathbf{w}(q)\dot{q} = 0 \Rightarrow \ddot{\mathbf{w}}(\phi)\mathbf{V} = 0$

3. Constrained velocities $\mathbf{V} = w_v(\phi)v + w_u(\phi)u$, where $\ddot{\mathbf{w}}_u = \ddot{\mathbf{w}}_u = 0$

4. Reduced equations:

   $\dot{v} = f_v(\phi)v^2 + f_u(\phi)u^2 + f_{uv}(\phi)uv + f_\tau(\phi)\tau$

   $\dot{u} = g_v(\phi)v^2 + g_u(\phi)u^2 + g_{uv}(\phi)uv + g_\tau(\phi)\tau$

   $\dot{\phi} = h(\phi)u$
Twistcar - Analysis

Reduced formulation:

1. Normalize time by $1/\omega$, length by $l_1+l_2$, mass by $m$ 
   two nondimensional parameters: $\alpha = \frac{l_3}{l_1+l_2}$, $\beta = \frac{l_1}{l_1+l_2}$

2. Body velocities $V = (v_x, v_y, \dot{\theta}, \dot{\phi})^T$, constraints $\dot{w}(q)q = 0 \Rightarrow \ddot{w}(\phi)V = 0$

3. Constrained velocities $V = w_v(\phi)v + w_u(\phi)u$, where $\ddot{w}_u = \ddot{w}_v = 0$

 motion along $w_v$

$\phi = \text{const.}$

 motion along $w_u$

$\dot{\phi} \neq 0$. 
Twistcar - Analysis

Reduced formulation:

1. Normalize time by $1/\omega$, length by $l_1+l_2$, mass by $m$

   two nondimensional parameters: $\alpha = \frac{l_3}{l_1+l_2}$, $\beta = \frac{l_1}{l_1+l_2}$

2. Body velocities $\mathbf{v} = (v_x, v_y, \dot{\theta}, \dot{\phi})^T$, constraints $\mathbf{w}(q)\dot{q} = 0 \Rightarrow \tilde{\mathbf{W}}(\phi)\mathbf{V} = 0$

3. Constrained velocities $\mathbf{V} = \mathbf{w}_v(\phi)v + \mathbf{w}_u(\phi)u$, where $\tilde{\mathbf{W}}_v = \tilde{\mathbf{W}}_u = 0$

4. Reduced equations:

   $\dot{v} = f_v(\phi)v^2 + f_u(\phi)u^2 + f_{uv}(\phi)uv + f_{\tau}(\phi)\tau$
   $\dot{u} = g_v(\phi)v^2 + g_u(\phi)u^2 + g_{uv}(\phi)uv + g_{\tau}(\phi)\tau$
   $\dot{\phi} = h(\phi)u$

5. Angle input: $\phi(t), \dot{\phi}(t), \ddot{\phi}(t) \rightarrow u(t), \dot{u}(t)$ known, eliminate $\tau$

   $\Rightarrow$ single linear time-varying ODE (integrable):

   $\dot{v} = F\left(\phi(t), \dot{\phi}(t)\right)v + G\left(\phi(t), \dot{\phi}(t), \ddot{\phi}(t)\right)$
Perturbation expansion

Angle input: $\phi(t) = \varepsilon \sin t$, where $\varepsilon \ll 1$

Assume a solution of the form $v(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + ...$

Expand $\dot{v} = F(\phi, \dot{\phi})v + G(\phi, \dot{\phi}, \ddot{\phi})$ as series in $\phi, \dot{\phi}, \ddot{\phi}$, then of $\varepsilon$

Solution: $v(t) = v_0 + \varepsilon^2 \left( \frac{\alpha \beta (\alpha + \beta)}{2(1 - \alpha)^3} t + A_1(\alpha, \beta) \sin(2t) + A_2(\alpha, \beta) v_0 (1 - \cos(2t)) \right) + O(\varepsilon^3)$

Acceleration is monotonic in $\alpha, \beta$, vanishes at $O(\varepsilon^2)$ for $\beta = 0$.

Plug solution into $\dot{\phi} = h(\phi)u$, then $V = w_u(\phi)v + w_v(\phi)u$, expand...
Perturbation expansion

Angle input: $\phi(t) = \varepsilon \sin t$, where $\varepsilon \ll 1$

Assume a solution of the form $v(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + ...$

Expand $\dot{v} = F(\phi, \dot{\phi})v + G(\phi, \dot{\phi}, \ddot{\phi})$ as series in $\phi, \dot{\phi}, \ddot{\phi}$, then of $\varepsilon$

Solution: $v(t) = v_0 + \varepsilon^2 \left( \frac{\alpha \beta (\alpha + \beta)}{2(1 - \alpha)^3} - A_1(\alpha, \beta) \sin(2t) + A_2(\alpha, \beta) v_0 (1 - \cos(2t)) \right) + \mathcal{O}(\varepsilon^3)$

Acceleration is monotonic in $\alpha, \beta$, vanishes at $\mathcal{O}(\varepsilon^2)$ for $\beta = 0$.

Plug solution into $\dot{\phi} = h(\phi)u$, then $V = w_u(\phi)v + w_v(\phi)u$, expand...

$$\theta(t) = \varepsilon \left( v_0 (1 - \cos t) - \frac{\alpha}{\alpha - 1} \sin t \right) + \varepsilon^3 \left[ B_0(\alpha, \beta) t \cos t + B_1(\alpha, \beta) \sin t + B_2(\alpha, \beta) \sin(3t) + B_3(\alpha, \beta) v_0 (8 - 9 \cos(t) + \cos(3t)) \right] + \mathcal{O}(\varepsilon^4)$$

$\Rightarrow \mathcal{O}(\varepsilon^3)$ divergence of body orientation angle
Roller-Racer motion reversal

Angle input: \( \phi(t) = \pi + \epsilon \sin t \), where \( \epsilon \ll 1 \). Expand, solve...

\[
v_x(t) = -(\alpha + 1)v_0 + \epsilon^2 \left[ C_0(\alpha, \beta)v_0 + \frac{\alpha \beta(\alpha - \beta)}{2(\alpha + 1)^2} t \right] + C_1(\alpha, \beta)\sin(2t) + C_2(\alpha, \beta)v_0 \cos(2t) + O(\epsilon^3)
\]

Direction of acceleration \( a \) depends on \( \text{sgn}(\alpha - \beta) \):

Non-degenerate optimum of \( |a| \):

For given \( \beta \): \( \alpha = \frac{\beta}{\beta + 2} \)

For given \( \alpha \): \( \beta = \frac{\alpha}{2} \)
Torque input - analysis

Reduced state equations:
\[
\begin{align*}
\dot{v} &= f_v(\phi)v^2 + f_u(\phi)u^2 + f_{uv}(\phi)uv + f_\tau(\phi)\tau \\
\dot{u} &= g_v(\phi)v^2 + g_u(\phi)u^2 + g_{uv}(\phi)uv + g_\tau(\phi)\tau \\
\dot{\phi} &= h(\phi)u
\end{align*}
\]

Perturbation expansion:
\[
\begin{align*}
v(t) &= v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + O(\varepsilon^3) \\
u(t) &= \varepsilon u_1(t) + \varepsilon^2 u_2(t) + O(\varepsilon^3) \\
\phi(t) &= \varepsilon \phi_1(t) + \varepsilon^2 \phi_2(t) + O(\varepsilon^3)
\end{align*}
\]

Zero-order dynamics: \( \dot{v}_0(t) = 0 \quad \Rightarrow \quad v_0 = \text{const.} \)

First-order dynamics:
\[
\begin{bmatrix}
\dot{v}_1 \\
\dot{u}_1 \\
\dot{\phi}_1
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & \alpha^2 + \alpha \beta - \alpha - \beta & \frac{1}{\alpha \beta} v_0 \\
0 & -(\alpha - 1)^2 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
u_1 \\
\phi_1
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-\frac{1}{\alpha^2 \beta^2} \\
0
\end{bmatrix} \sin(t)
\]

Solution contains transient terms of \( e^{\lambda_i t} + \text{terms of } \sin(t+\psi) \)

Negative real eigenvalues \( \lambda_{1,2} \) and \( \lambda_3 = 0 \quad \Rightarrow \text{decay to steady-state} \)
(neutral stability of free system due to energy conservation)
Steady-state solution:

\[
v(t) = v_0 + \varepsilon^2 \left[ \frac{(1 - \alpha)(\alpha + \beta)}{2\alpha\beta((\alpha - 1)^2 v_0^2 + \alpha^2)((\alpha - 1)^2 v_0^2 + \beta^2)} t \right. \\
+ C_1(\alpha, \beta, v_0)(\cos(2t) - 1) + C_2(\alpha, \beta, v_0)\sin(2t) \left. \right] + O(\varepsilon^3)
\]

\[
\theta(t) = \varepsilon \left[ -\frac{(\alpha - 1)((\alpha^2 v_0^2 - 2\alpha v_0^2 - \alpha\beta + v_0^2)(\alpha - 1) + \alpha(\alpha^2 + \alpha\beta - \alpha - \beta))v_0}{\alpha\beta((\alpha - 1)^2 v_0^2 + \alpha^2)((\alpha - 1)^2 v_0^2 + \beta^2)} (\cos(t) - 1) \\
+ \frac{(\alpha - 1)((\alpha - 1)(\alpha^2 + \alpha\beta - \alpha - \beta)v_0^2 - \alpha(\alpha^2 v_0^2 - 2\alpha v_0^2 - \alpha\beta + v_0^2))}{\alpha\beta((\alpha - 1)^2 v_0^2 + \alpha^2)((\alpha - 1)^2 v_0^2 + \beta^2)} \sin(t) \right] + O(\varepsilon^2)
\]

"Initial" speed \(v_0\) should be updated - \(\bar{v}(t)\) grows with time

Oscillation amplitude of \(\theta(t)\) depends on \(v_0\)

\[\rightarrow \text{decays as } \frac{1}{\bar{v}(t)^2} \implies \text{straight line motion}\]
Roller-racer torque input

Torque \( \tau(t) = \varepsilon \sin t \), expansion about \( \phi = \pi \)

First-order dynamics:

\[
\begin{bmatrix}
\dot{v}_1 \\
\dot{u}_1 \\
\dot{\phi}_1
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{\alpha^2 - \alpha \beta + \alpha - \beta}{\alpha \beta} & -\frac{1}{\alpha \beta} v_0 \\
0 & -(\alpha + 1)^2 & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
u_0 \\
\phi_1
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-\frac{1}{\alpha^2 \beta^2} \\
0
\end{bmatrix} \varepsilon \sin(t)
\]

Characteristic polynomial of linearization matrix:

\[
\lambda^3 - \frac{\alpha^2 + \alpha \beta - \alpha - \beta}{\alpha \beta} v_0 \lambda^2 - \frac{(\alpha + 1)^2}{\alpha \beta} v_0^2 \lambda = 0
\]

\[\Rightarrow \phi = \pi \text{ is unstable for any sign of } v_0\]
Preliminary experiments (movies)

VEX robotics kit.
Steering angle input by servo motor, triangular wave.

1. Twistcar
2. Roller-Racer backwards
3. Roller-Racer reversal
Future extensions

Improved experiments - measurements, torque control
Effects of wheels slippage → hybrid dynamics
stick-slip transitions (3D model, effect of c.o.m. height)
Control – path stabilization (hybrid control law?)
Outline

1. The Twistcar toy (Ofir Chakon, MSc work)

2. Microswimmers (Emiliya Gutman, PhD work)

3. Others: slippage effects on legged locomotion
   senior project - reverse a truck & trailer
Microswimmers – intro

Swimming microorganisms [Taylor, 1951]
Low Reynolds num. hydrodynamics – viscosity \(\gg\) inertia
Simplistic theoretical models:
3-spheres, pushmepullyou, Purcell’s 3-link swimmer...
Studied nonlinear controllability, geometric mechanics
Swimming microorganisms [Taylor, 1951]
Low Reynolds num. hydrodynamics – viscosity $\gg$ inertia
Simplistic theoretical models:
3-spheres, pushmepullyou, Purcell’s 3-link swimmer...
Studied nonlinear controllability, geometric mechanics
All assume: unbounded fluid, kinematic shape control

**My works:** wall effects, torque input, elastic tail, robot
Magnetic microswimmers

For biomedical applications, magnetic actuation

Helical rigid tail, rotating magnetic field:

Zhang et al 2009
Ghosh and Fischer 2009

Flexible tail, rotating/oscillating magnetic field:

Dreyfus et al 2005
Gao et al 2010
Pak et al 2011

Nelson’s planar undulating 2-link and 3-link magnetic swimmer

Optimal frequency \( \omega \) for speed \( V \) or scaled \( V/\omega \sim \) displacement \( X \)
Magnetic microswimmers

Optimal frequency $\omega$ for speed $V$ or scaled $V/\omega \sim$ displacement $X$
Planar undulating two-link model

Coordinates of the swimmer: \( \mathbf{q} = (\mathbf{q}_b^T, \phi)^T \)
where \( \mathbf{q}_b = (x, y, \theta)^T \) – body position,
\( \phi \) – swimmer's shape
Passive torsion spring at the joint: \( \tau = -k\phi \)
(\( \tau \) - torque, \( k \) - torsional stiffness)

Two magnetic dipoles: \( h_1, h_2 \) aligned with the links' longitudinal axes \( \mathbf{t}_i \)
External magnetic field: \( \mathbf{B}(t) = B_x \left[ 1, \varepsilon \sin(\omega t) \right]^T \)
Aligning magnetic torque on the \( i^{th} \) link: \( \mathbf{M}_i = h_i \mathbf{t}_i \times \mathbf{B} \)

Stokes Equations \((Re=0)\): \( \nabla \cdot \mathbf{u} = 0, \quad -\nabla p + \mu \nabla^2 \mathbf{u} = 0 \)
Linear relation between viscous forces and velocities (resistance)
Dynamics – formulation (movie)

Resistive Force Theory (Cox 1970, Gray & Hancock 1955):

\[ f_i = -c_i(t(v_i \cdot t_i)) - c_n(t(v_i \cdot n_i)) \]

Kinematic relations – express velocities in terms of Quasistatic motion – each link is in force+torque balance

Dynamic equations:

Simulation example:

\[ \omega_i^{(n+1)} = 2 \ln(\omega_i^{(n)}) \]

\[ l_{cc} c_l a_q \]

\[ q_{A q C q B} D_{q E B} \]

\[ (0) = 0, l = 1, k = 0.3, h_1 = 0, h_2 = 1, B = 1, \varepsilon = 0.4 \text{ and } \omega = 1 \]
Analysis

Two characteristic time scales:

1. Visco-magnetic time – magnetic alignment of a single link

\[ h_1 = 0, h_2 \neq 0, \mathbf{B} = \begin{bmatrix} B_x \\ 0 \end{bmatrix}, \phi = 0 \Rightarrow \dot{\theta} = -\frac{1}{t_m} \sin \theta \Rightarrow \theta \approx \theta_0 e^{-t/t_m}, \quad t_m = \frac{4 c_I l^3}{3 B_x h_2} \]

2. Visco-elastic time – alignment of elastic joint

\[ h_1 = 0, h_2 = 0, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, k > 0 \Rightarrow \dot{\phi} = -\frac{1}{2 t_k} \frac{3 + \cos \phi}{3 - \cos \phi} \phi \approx \phi_0 e^{-t/t_k}, \quad t_k = \frac{c_I l^3}{12 k} \]

Scale time by \( t_m \),
define two non-dimensional parameters: \( \alpha = \frac{t_m}{t_k}, \beta = \frac{h_1}{h_2} \)

Three cases:

I. \( h_1 = 0, h_2 \neq 0, k \neq 0 \Rightarrow \beta = 0 \)
II. \( h_1 \neq 0, h_2 \neq 0, k = 0 \Rightarrow \alpha = 0 \)
III. \( h_1 \neq 0, h_2 \neq 0, k \neq 0 \Rightarrow \alpha \neq 0, \beta \neq 0 \)
Perturbation expansion

Assume small amplitude oscillations of \( B_y(t) \), \( \varepsilon \ll 1 \)

Expand \( q(t) = \varepsilon q_1(t) + \varepsilon^2 q_2(t) + \varepsilon^3 q_3(t) + ... \)

Dynamics depend only on angles \( \theta, \phi \). First-order system:

\[
\begin{bmatrix}
    \dot{\theta}_1(t) \\
    \dot{\phi}_1(t)
\end{bmatrix} =
\begin{bmatrix}
    -5\beta + 3 & 0.5\alpha + 3 \\
    8\beta - 8 & -\alpha - 8
\end{bmatrix}
\begin{bmatrix}
    \theta_1(t) \\
    \phi_1(t)
\end{bmatrix} +
\begin{bmatrix}
    5\beta - 3 \\
    -8\beta + 8
\end{bmatrix}\sin(\omega t)
\]

Solution is of the form: \( C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 \sin(\omega t + \varphi) \)

Stability conditions: \( \text{Re}(\lambda_1, \lambda_2) < 0 \rightarrow \alpha + 5\beta + 5 > 0 \) and \( \alpha + 16\beta + \alpha\beta > 0 \)

Case I: \( \beta = 0 \Rightarrow \alpha > 0 \); Case II: \( \alpha = 0 \Rightarrow \beta > 0 \)

Case III: \( \alpha, \beta \neq 0 \Rightarrow \alpha \) can be negative \( \Rightarrow \) destabilizing torsion spring \( \beta \) can be negative \( \Rightarrow \) opposite links' magnetization
Perturbation expansion

Stability conditions: \( \text{Re}(\lambda_1, \lambda_2) < 0 \rightarrow \alpha + 5\beta + 5 > 0 \) and \( \alpha + 16\beta + \alpha\beta > 0 \)

Case I: \( \beta = 0 \Rightarrow \alpha > 0 \); Case II: \( \alpha = 0 \Rightarrow \beta > 0 \)

Case III: \( \alpha, \beta \neq 0 \Rightarrow \alpha \) can be negative \( \Rightarrow \) destabilizing torsion spring
\( \beta \) can be negative \( \Rightarrow \) opposite links' magnetization
Leading-order solution

The second order equation for the forward motion $x(t)$:

$$\ddot{x}_2 = \frac{l}{2} \left[ -(\beta + 1)\theta_{(1)}^2 - \left( \frac{1}{4} \alpha + 3 \right) \phi_{(1)}^2 + (\beta - 4)\theta_{(1)}\phi_{(1)} + ((\beta + 1)\theta_{(1)} + (-\beta + 3)\phi_{(1)}) \sin(\omega t) \right]$$

The steady state solution is of the form:

$$x_{(2)}(t) = M(\omega, \alpha, \beta) \sin(2\omega t + \varphi(\omega, \alpha, \beta)) + \ddot{V}(\omega, \alpha, \beta)t \quad \text{Net forward motion}$$

The average speed and the net displacement per cycle (leading-order):

\[
\begin{align*}
X &= \frac{2\pi \omega (1 - \beta)(\alpha + 16\beta + \alpha\beta)}{\omega^4 + (\alpha^2 + 8\alpha\beta + 8\alpha + 25\beta^2 + 18\beta + 25)\omega^2 + \alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2 + 32\alpha\beta^2 + 32\alpha\beta + 256\beta^2} \\
V &= \frac{X}{2\pi / \omega} = \\
&= \frac{\omega^2 (1 - \beta)(\alpha + 16\beta + \alpha\beta)}{\omega^4 + (\alpha^2 + 8\alpha\beta + 8\alpha + 25\beta^2 + 18\beta + 25)\omega^2 + \alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2 + 32\alpha\beta^2 + 32\alpha\beta + 256\beta^2}
\end{align*}
\]
Optimal actuation frequency

For $X$ the optimal frequency $\omega_x$ is:

$$\omega_x = \frac{1}{\sqrt{6}} \left( -\left( \alpha^2 + 8\alpha \beta + 8\alpha + 25\beta^2 + 18\beta + 25 \right)^{\frac{1}{3}} + \left( \alpha^2 + 8\alpha \beta + 8\alpha + 25\beta^2 + 18\beta + 25 \right)^{\frac{2}{3}} + 12\left( \alpha^2 \beta^2 + 2\alpha^2 \beta + \alpha^2 + 32\alpha \beta^2 + 32\alpha \beta + 256 \beta^2 \right) \right)^{\frac{1}{3}}$$

For $V$ the optimal frequency $\omega_v$ is:

$$\omega_v = \sqrt{\alpha + 16\beta + \alpha \beta}$$

$l=1$, $k=0.3$, $h_1=0$, $h_2=1$, $B_x=1$, $\varepsilon=0.4$ and $\omega=1$
Optimal swimmer’s parameters

- Case I ($\beta=0$) the optimal value of $V$ is: $V\left(\omega_v = \sqrt{5}, \alpha_v = 5\right) = 0.05\epsilon^2 \frac{l}{t_m}$
- Case II ($\alpha=0$) the optimal value of $V$ is: $V\left(\omega_v = \frac{4}{\sqrt{3}}, \beta_v = \frac{1}{3}\right) = 0.08\epsilon^2 \frac{l}{t_m}$
Optimal swimmer’s parameters

- For case III ($\alpha, \beta \neq 0$) the optimal value of $V$ is:

$$V\left(\omega_v = 1.81922, \alpha_v = -2.7, \beta_v = 0.45197\right) = 0.0873\varepsilon^2 \frac{l}{t_m}$$

Optimum is obtained at $\alpha < 0 \rightarrow$ destabilizing spring
Destabilizing “torsion spring”

\[ \tau(\phi) = -\frac{1}{2} \tilde{k} b (l_0 - 2b) \phi + O(\phi^3) \]

- \( l_0 < 2b \) \( \phi = 0 \) is unstable (spring in tension)
- \( l_0 > 2b \) \( \phi = 0 \) is stable (spring in compression)

Can one manufacture this in a real micron-size swimmer??
Comparison to real swimmers

<table>
<thead>
<tr>
<th>Microswimmer</th>
<th>L (μm)</th>
<th>ε = by / bx</th>
<th>X* / ε²L</th>
<th>fₓ (Hz)</th>
<th>V* / ε²L₀fₓ</th>
<th>fᵥ (Hz)</th>
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<tbody>
<tr>
<td>Dreyfus et al [1]</td>
<td>24</td>
<td>10.3/8.9 = 1.16</td>
<td>0.068</td>
<td>10</td>
<td>0.031</td>
<td>4</td>
</tr>
<tr>
<td>Pak et al [2]</td>
<td>5.8</td>
<td>10/9.5 = 1.05</td>
<td>0.149</td>
<td>15</td>
<td>0.093</td>
<td>35</td>
</tr>
<tr>
<td>Our model</td>
<td>2l</td>
<td></td>
<td>0.31</td>
<td></td>
<td>ωₓ → 0</td>
<td>0.15</td>
</tr>
</tbody>
</table>


Reported experimental values of maximal displacement and speed are below the optimal values according to our theoretical model, yet they are in the same order of magnitude.

**Current work with B. Nelson at ETH:**
Match time scales tₘ, tₖ to experimental microswimmer prototype
Test the theoretical predictions of optimal frequency + parameters

What’s the relation of all this to CONTROL???
Outline

1. The Twistcar toy (Ofir Chakon, MSc work)

2. Microswimmers (Emiliya Gutman, PhD work)

3. Others: slippage effects on legged locomotion
   senior project - reverse a truck & trailer
Slippage in legged locomotion

Benny Gamus, Moti Moravia (MSc works)

Dynamic legged locomotion – nonlinear hybrid system
Control of legged robots – widely studied
Passive dynamic walking – classical models (rigid links)

All models assume sticking contact at feet (high friction)

Our works: studied the effects of slippage under Coulomb friction model on performance (stability, speed, energetic efficiency)
Reverse of LEGO truck & trailer

Senior thesis project of Iddo Perel and Hallel Bunis

Stabilize truck & trailer(s) in reverse

Nonholonomic, driftless nonlinear control system

Linearization about straight line, control input is $\omega/v$

State feedback / nested control loops

Implementation in LEGO NXT, controlled by Simulink
Reverse of LEGO truck & trailer

movie
Thank You
תודה
Questions??
questions??