Cramer-Rao Bound

What is the Cramer-Rao Lower Bound (CRLB) and What Does it Mean?

According to Bar Shalom*

- **"The mean square error corresponding to the estimator of a parameter cannot be smaller than a certain quantity related to the likelihood function"**
- **"If an estimator's variance is equal to the CRLB, then the estimator is called efficient"**

Formula for CRLB found in many texts

$$E\left(\left[\hat{x}(Z)-x_0\right]\left[\hat{x}(Z)-x_0\right]^T\right)\geq J^{-1}$$

$$J=E\left(\left[\nabla_x \ln\Lambda(x)\right]\left[\nabla_x \ln\Lambda(x)\right]^T\right)_{x=x_0}$$

What does this mean and how do I program it?

What does Zarchan say about the utility of the CRLB?

"If an estimator's variance is equal to the CRLB, then perhaps the estimator is called not practical"

*Bar-Shalom, Y., Li,X. and Kirubarajan, T., "Estimation With Applications to Tracking and Navigation, Theory Algorithms And Software," John Wiley & Sons, Inc., New York, 2001, pp 109-110.

Cramer-Rao Lower Bound (CRLB) as an Algorithm

It can be shown* in a more understandable way that according to the CRLB the best a least squares filter can do is given by

$$P_k^{-1} = \left(\Phi P_{k-1}\Phi^T\right)^{-1} + H^T R^{-1} H$$

Where P is the covariance matrix, Φ is the fundamental matrix, H is the measurement matrix and R is the measurement noise matrix. P represents the smallest error in the estimate that is possible. The above equation can be improved slightly to make it easier to program. Let

$$A_k = P_k^{-1}$$

Since

$$\left(\Phi P_{k-1}\Phi^T\right)^{-1} = \left(\Phi^T\right)^{-1} P_{k-1}^{-1}\Phi^{-1} = \left(\Phi^T\right)^{-1} A_{k-1}\Phi^{-1}$$

Therefore the the CRLB equation can be rewritten as

$$A_k = \left(\Phi^T\right)^{-1} A_{k-1}\Phi^{-1} + H^T R^{-1} H$$

The initial condition on the preceding matrix difference equation is

$$A_0 = 0$$

*Taylor, J.H., “The Cramer-Rao Estimation Error Lower Bound Computation for Deterministic Nonlinear Systems,” IEEE Transactions On Automatic Control, Vol. AC-24, No. 2, April 1979, pp 343-344.

How Does the Cramer-Rao Lower Bound (CRLB) Relate to Our Recursive Least Squares Filter?

- We have studied recursive least squares filters and found their gains and formulas predicting their performance.
- We know that a linear polynomial Kalman filter with zero process noise and infinite initial covariance matrix is identical to the recursive least squares filter.
- The recursive least squares filter also represents the best a filter can do.

Does the CRLB formula yield the same answers as can be obtained by examining the covariance matrix of the recursive least squares filter?

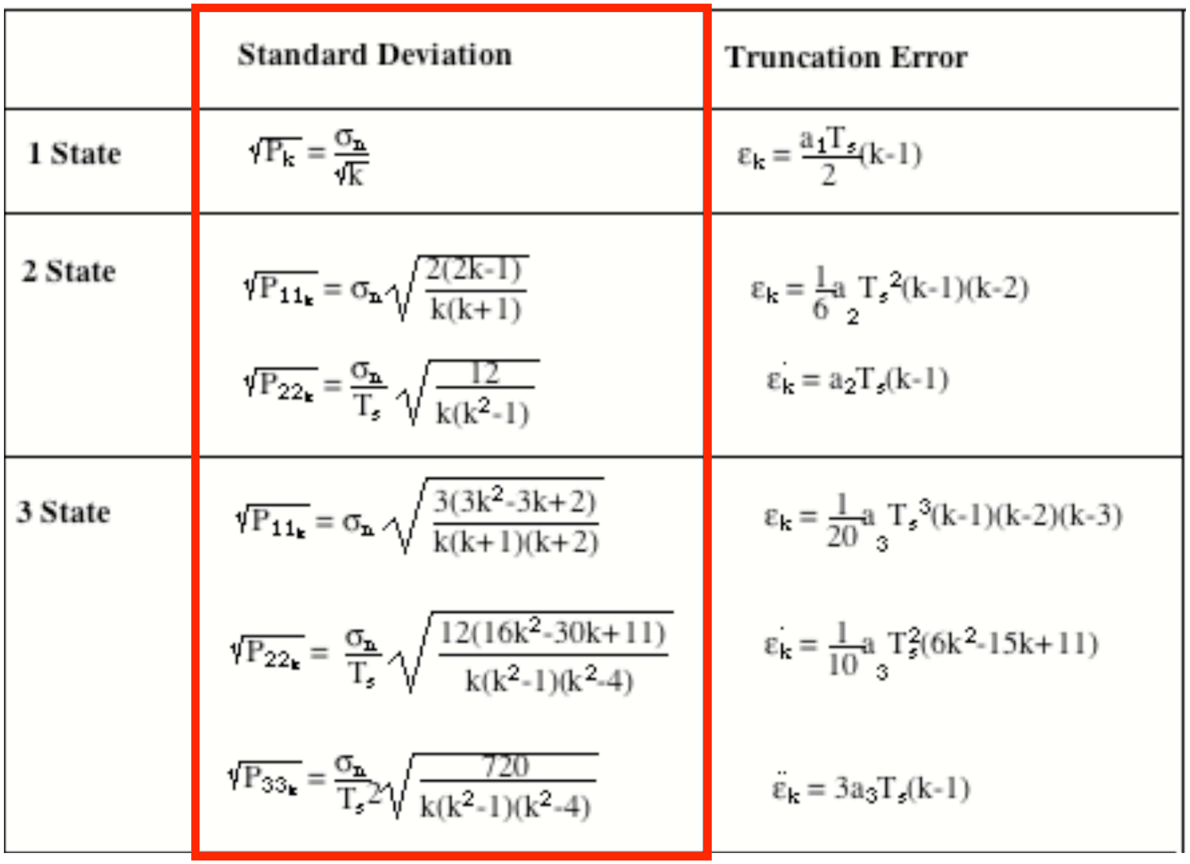
Recall Recursive Least Squares Filter Structure and Gains For Different Order Systems

	Filter	Gains
1 State	$Res_k = x_k^* - \hat{x}_{k-1}$ $\hat{x}_k = \hat{x}_{k-1} + K_{1_k} Res_k$	$K_{1_k} = \frac{1}{k}$
2 State	$Res_k = x_k^* - \hat{x}_{k-1} - \hat{\dot{x}}_{k-1} T_s$ $\hat{x}_k = \hat{x}_{k-1} + \hat{\dot{x}}_{k-1} T_s + K_{1_k} Res_k$ $\hat{\dot{x}}_k = \hat{\dot{x}}_{k-1} + K_{2_k} Res_k$	$K_{1_k} = \frac{2(2k-1)}{k(k+1)}$ $K_{2_k} = \frac{6}{k(k+1)T_s}$
3 State	$Res_k = x_k^* - \hat{x}_{k-1} - \hat{\dot{x}}_{k-1} T_s - .5\hat{\ddot{x}}_{k-1} T_s^2$ $\hat{x}_k = \hat{x}_{k-1} + \hat{\dot{x}}_{k-1} T_s + .5\hat{\ddot{x}}_{k-1} T_s^2 + K_{1_k} Res_k$ $\hat{\dot{x}}_k = \hat{\dot{x}}_{k-1} + \hat{\ddot{x}}_{k-1} T_s^2 + K_{2_k} Res_k$ $\hat{\ddot{x}}_k = \hat{\ddot{x}}_{k-1} + K_{3_k} Res_k$	$K_{1_k} = \frac{3(3k^2-3k+2)}{k(k+1)(k+2)}$ $K_{2_k} = \frac{18(2k-1)}{k(k+1)(k+2)T_s}$ $K_{3_k} = \frac{60}{k(k+1)(k+2)T_s^2}$

k=1,2,3,....

Note that the above Table tells us directly how to build the filter

Recall Formulas For Errors in Estimates of Different Order Recursive Least Squares Filters



	Standard Deviation	Truncation Error
1 State	$\sqrt{P_k} = \frac{\sigma_n}{\sqrt{k}}$	$\varepsilon_k = \frac{a_1 T_s}{2}(k-1)$
2 State	$\sqrt{P_{11_k}} = \sigma_n \sqrt{\frac{2(2k-1)}{k(k+1)}}$ $\sqrt{P_{22_k}} = \frac{\sigma_n}{T_s} \sqrt{\frac{12}{k(k^2-1)}}$	$\varepsilon_k = \frac{1}{6} a_2 T_s^2 (k-1)(k-2)$ $\dot{\varepsilon}_k = a_2 T_s (k-1)$
3 State	$\sqrt{P_{11_k}} = \sigma_n \sqrt{\frac{3(3k^2-3k+2)}{k(k+1)(k+2)}}$ $\sqrt{P_{22_k}} = \frac{\sigma_n}{T_s} \sqrt{\frac{12(16k^2-30k+11)}{k(k^2-1)(k^2-4)}}$ $\sqrt{P_{33_k}} = \frac{\sigma_n}{T_s^2} \sqrt{\frac{720}{k(k^2-1)(k^2-4)}}$	$\varepsilon_k = \frac{1}{20} a_3 T_s^3 (k-1)(k-2)(k-3)$ $\dot{\varepsilon}_k = \frac{1}{10} a_3 T_s^2 (6k^2-15k+11)$ $\ddot{\varepsilon}_k = 3 a_3 T_s (k-1)$

k=1,2,3,….

Note that the covariance expressions in the above Table tells us directly the best the filter can perform

Important Matrices for Different Order Linear Polynomial Kalman Filters

States	Order	Systmes Dynamics	Fundamental	Measurement	Noise
1	0	$F=1$	$\Phi_k=1$	$H=1$	$R_k = \sigma_n^2$
2	1	$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\Phi_k = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}$	$H = [1 \quad 0]$	$R_k = \sigma_n^2$
3	2	$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\Phi_k = \begin{bmatrix} 1 & T_s & .5T_s^2 \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix}$	$H = [1 \quad 0 \quad 0]$	$R_k = \sigma_n^2$

Above matrices will be used in the CRLB equation

One-State Example

From previous slide we see that for a one-state system

$$\Phi = 1,\ R = \sigma^2 \ \ and\ \ H = 1$$

Therefore from the formula for the CRLB we can say that

$$A_k = \left(\Phi^T\right)^{-1} A_{k-1}\Phi^{-1} + H^T R^{-1} H = A_{k-1} + \frac{1}{\sigma^2}$$

With initial condition

$$A_0 = 0$$

Therefore by inspection we can see that

$$A_1 = A_0 + \frac{1}{\sigma^2} = 0 + \frac{1}{\sigma^2} = \frac{1}{\sigma^2}$$

$$A_2 = A_1 + \frac{1}{\sigma^2} = \frac{1}{\sigma^2} + \frac{1}{\sigma^2} = \frac{2}{\sigma^2}$$

$$A_3 = A_2 + \frac{1}{\sigma^2} = \frac{2}{\sigma^2} + \frac{1}{\sigma^2} = \frac{3}{\sigma^2}$$

By induction it becomes apparent that

$$A_k = \frac{k}{\sigma^2}$$

Which means that

$$P_k = \frac{\sigma^2}{k}$$

Since $P_k = A_k^{-1}$

Therefore CRLB covariance for one-state system is identical to formula for one-state covariance of recursive least squares filter!

Two-State Example

From “Important Matrices” slide we see that for a two-state system

$$\Phi = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \qquad R = \sigma^2 \qquad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Therefore from the formula for CRLB we can say that

$$A_k = \left(\Phi^T\right)^{-1} A_{k-1} \Phi^{-1} + H^T R^{-1} H$$

With initial condition

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$P_k = A_k^{-1}$$

We will use a computer simulation (with a matrix inverse routine) to compute the diagonal elements of the covariance matrix of CRLB. This method will be compared to a Kalman filter with zero process noise and infinite initial covariance matrix and to the formulas for the recursive least squares filter

Listing For CRLB Comparison in 2-State System-1

```
        IMPLICIT REAL*8(A-H,O-Z)
        REAL*8 M(2,2),P(2,2),K(2,1),PHI(2,2),H(1,2),R(1,1),PHIT(2,2)
        REAL*8 PHIP(2,2),HT(2,1),KH(2,2),IKH(2,2),A(2,2)
        REAL*8 MHT(2,1),HMHT(1,1),HMHTR(1,1),HMHTRINV(1,1),IDN(2,2)
        REAL*8 PHIINV(2,2),PHITINV(2,2),PHITINVA(2,2),TEMP1(2,2)
        REAL*8 PP(2,2)
        REAL*8 K1GM,K2GM,K3GM
        INTEGER ORDER
        OPEN(1,STATUS='UNKNOWN',FILE='DATFIL')
        ORDER=2
        TS=.5
        SIGNOISE=3.
        DO 1000 I=1,ORDER
        DO 1000 J=1,ORDER
                PHI(I,J)=0.
                P(I,J)=0.
                IDN(I,J)=0.
                A(I,J)=0.
1000    CONTINUE
        IDN(1,1)=1.
        IDN(2,2)=1.
        P(1,1)=99999999999999.
        P(2,2)=99999999999999.
        PHI(1,1)=1
        PHI(1,2)=TS
        PHI(2,2)=1
        DO 1100 I=1,ORDER
                H(1,I)=0.
1100    CONTINUE
        H(1,1)=1
        CALL MATTRN(H,1,ORDER,HT)
        R(1,1)=SIGNOISE**2
        CALL MATTRN(PHI,ORDER,ORDER,PHIT)
        CALL MTINV(PHI,ORDER,PHIINV)
        CALL MTINV(PHIT,ORDER,PHITINV)
```

Inputs for comparison

A, I, P, Φ and H matrices for 2-state system

H^T, R, Φ^T, Φ^{-1}, $(\Phi^T)^{-1}$ matrices

Listing For CRLB Comparison in 2-State System -2

```
              DO 10 XN=1.,100.
              CALL MATMUL(PHI,ORDER,ORDER,P,ORDER,ORDER,PHIP)
              CALL MATMUL(PHIP,ORDER,ORDER,PHIT,ORDER,ORDER,M)
              CALL MATMUL(M,ORDER,ORDER,HT,ORDER,1,MHT)
              CALL MATMUL(H,1,ORDER,MHT,ORDER,1,HMHT)
              HMHTR(1,1)=HMHT(1,1)+R(1,1)
              HMHTRINV(1,1)=1./HMHTR(1,1)
              CALL MATMUL(MHT,ORDER,1,HMHTRINV,1,1,K)
              CALL MATMUL(K,ORDER,1,H,1,ORDER,KH)
              CALL MATSUB(IDN,ORDER,ORDER,KH,IKH)
              CALL MATMUL(IKH,ORDER,ORDER,M,ORDER,ORDER,P)
              IF(XN<2)THEN
                      P11GM=9999999999.
                      P22GM=9999999999.
              ELSE
                      P11GM=2.*(2.*XN-1)*SIGNOISE*SIGNOISE/
     1                        (XN*(XN+1.))
                      P22GM=12.*SIGNOISE*SIGNOISE/(XN*(XN*XN-1.)
     1                        *TS*TS)
              ENDIF
              CALL MATMUL(PHITINV,ORDER,ORDER,A,ORDER,ORDER,PHITINVA)
              CALL MATMUL(PHITINVA,ORDER,ORDER,PHIINV,ORDER,ORDER,
     1                TEMP1)
              DO 1001 I=1,ORDER
              DO 1001 J=1,ORDER
              A(I,J)=TEMP1(I,J)
 1001         CONTINUE
              A(1,1)=TEMP1(1,1)+1./SIGNOISE**2
              CALL MTINV(A,ORDER,PP)
              WRITE(9,*)XN,P(1,1),P11GM,PP(1,1),P(2,2),P22GM,PP(2,2)
              WRITE(1,*)XN,P(1,1),P11GM,PP(1,1),P(2,2),P22GM,PP(2,2)
 10           CONTINUE
              CLOSE(1)
              PAUSE
              END
```

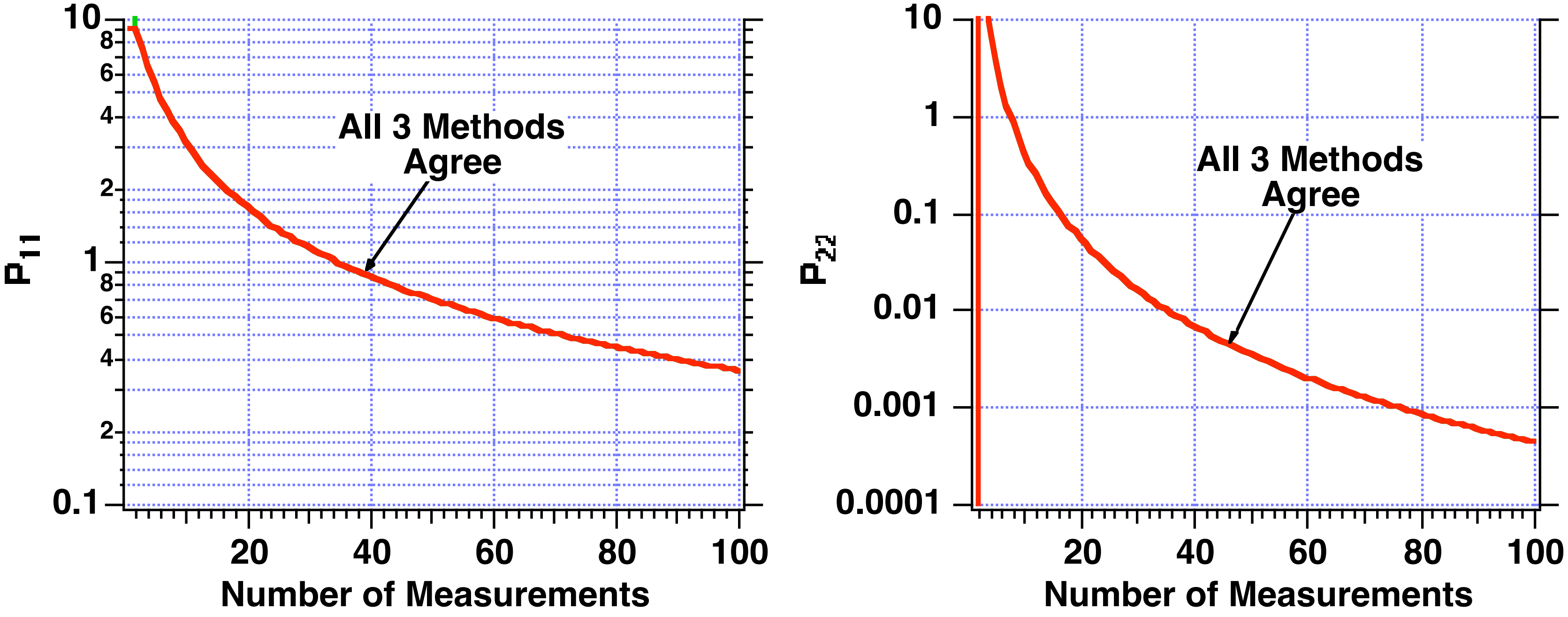
Ricatti equations with zero process noise for 2-state system

Recursive least squares filter for 2-state system

CRLB for 2-state system

Iterate 3 Methods

All Methods For Finding Best Performance in Two-State System Agree



CRLB and Kalman filter with zero process noise and infinite initial covariance matrix results are equivalent

Three-State Example

From “Important Matrices” slide we see that for a three-state system

$$\Phi = \begin{bmatrix} 1 & T_s & 0.5T_s^2 \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \sigma^2 \qquad H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Therefore from the formula for CRLB we can say that

$$A_k = \left(\Phi^T\right)^{-1} A_{k-1} \Phi^{-1} + H^T R^{-1} H$$

With initial condition

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P_k = A_k^{-1}$$

We will use a computer simulation (with a matrix inverse routine) to compute the diagonal elements of the covariance matrix of CRLB. This method will be compared to a Kalman filter with zero process noise and infinite initial covariance matrix and to the formulas for the recursive least squares filter

Listing For CRLB Comparison in 3-State System-1

```
        IMPLICIT REAL*8(A-H,O-Z)
        REAL*8 M(3,3),P(3,3),K(3,1),PHI(3,3),H(1,3),R(1,1),PHIT(3,3)
        REAL*8 PHIP(3,3),HT(3,1),KH(3,3),IKH(3,3),A(3,3)
        REAL*8 MHT(3,1),HMHT(1,1),HMHTR(1,1),HMHTRINV(1,1),IDN(3,3)
        REAL*8 PHIINV(3,3),PHITINV(3,3),PHITINVA(3,3),TEMP1(3,3)
        REAL*8 PP(3,3)
        REAL*8 K1GM,K2GM,K3GM
        INTEGER ORDER
        OPEN(1,STATUS='UNKNOWN',FILE='DATFIL')
        ORDER=3
        TS=.5
        SIGNOISE=3.
```

Inputs for comparison

```
        DO 1000 I=1,ORDER
        DO 1000 J=1,ORDER
                PHI(I,J)=0.
                P(I,J)=0.
                IDN(I,J)=0.
                A(I,J)=0.
1000    CONTINUE
        IDN(1,1)=1.
        IDN(2,2)=1.
        IDN(3,3)=1.
        P(1,1)=99999999999999.
        P(2,2)=99999999999999.
        P(3,3)=99999999999999.
        PHI(1,1)=1
        PHI(1,2)=TS
        PHI(1,3)=.5*TS*TS
        PHI(2,2)=1
        PHI(2,3)=TS
        PHI(3,3)=1.
        DO 1100 I=1,ORDER
                H(1,I)=0.
1100    CONTINUE
        H(1,1)=1
```

A, I, P, Φ and H matrices for 3-state system

```
        CALL MATTRN(H,1,ORDER,HT)
        R(1,1)=SIGNOISE**2
        CALL MATTRN(PHI,ORDER,ORDER,PHIT)
        CALL MTINV(PHI,ORDER,PHIINV)
        CALL MTINV(PHIT,ORDER,PHITINV)
```

H^T, R, Φ^T, Φ^{-1}, $(\Phi^T)^{-1}$ matrices

Listing For CRLB Comparison in 3-State System-2

```
        DO 10 XN=1.,100.
        CALL MATMUL(PHI,ORDER,ORDER,P,ORDER,ORDER,PHIP)
        CALL MATMUL(PHIP,ORDER,ORDER,PHIT,ORDER,ORDER,M)
        CALL MATMUL(M,ORDER,ORDER,HT,ORDER,1,MHT)
        CALL MATMUL(H,1,ORDER,MHT,ORDER,1,HMHT)
        HMHTR(1,1)=HMHT(1,1)+R(1,1)
        HMHTRINV(1,1)=1./HMHTR(1,1)
        CALL MATMUL(MHT,ORDER,1,HMHTRINV,1,1,K)
        CALL MATMUL(K,ORDER,1,H,1,ORDER,KH)
        CALL MATSUB(IDN,ORDER,ORDER,KH,IKH)
        CALL MATMUL(IKH,ORDER,ORDER,M,ORDER,ORDER,P)
        IF(XN<3)THEN
                P11GM=9999999999.
                P22GM=9999999999.
                P33GM=9999999999.
        ELSE
                P11GM=3.*(3.*XN*XN-3.*XN+2.)*SIGNOISE*SIGNOISE/
   1                    (XN*(XN+1.)*(XN+2.))
                P22GM=12.*(16.*XN*XN-30.*XN+11.)*SIGNOISE*
   1                    SIGNOISE/(XN*(XN*XN-1.)*(XN*XN-4.)
   1                    *TS*TS)
                P33GM=720.*SIGNOISE*SIGNOISE/(XN*(XN*XN-1.)*
   1                    (XN*XN-4.)*TS*TS*TS*TS)
        ENDIF
        CALL MATMUL(PHITINV,ORDER,ORDER,A,ORDER,ORDER,PHITINVA)
        CALL MATMUL(PHITINVA,ORDER,ORDER,PHIINV,ORDER,ORDER,
   1            TEMP1)
        DO 1001 I=1,ORDER
        DO 1001 J=1,ORDER
        A(I,J)=TEMP1(I,J)
 1001   CONTINUE
        A(1,1)=TEMP1(1,1)+1./SIGNOISE**2
        CALL MTINV(A,ORDER,PP)
        WRITE(9,*)XN,P(1,1),P11GM,PP(1,1),P(2,2),P22GM,PP(2,2),
   1            P(3,3),P33GM,PP(3,3)
        WRITE(1,*)XN,P(1,1),P11GM,PP(1,1),P(2,2),P22GM,PP(2,2),
   1            P(3,3),P33GM,PP(3,3)
 10     CONTINUE
        CLOSE(1)
        PAUSE
        END
```

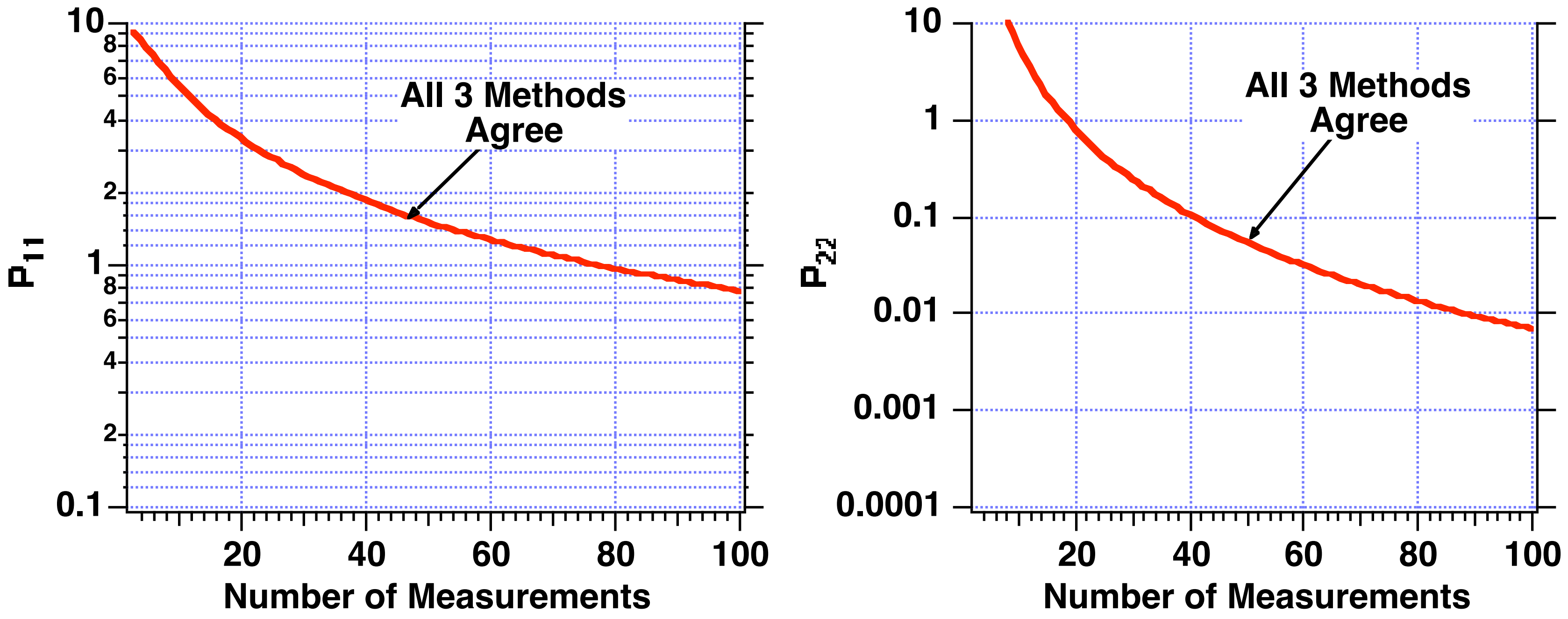
Ricatti equations with zero process noise for 3-state system

Recursive least squares filter for 3-state system

CRLB for 3-state system

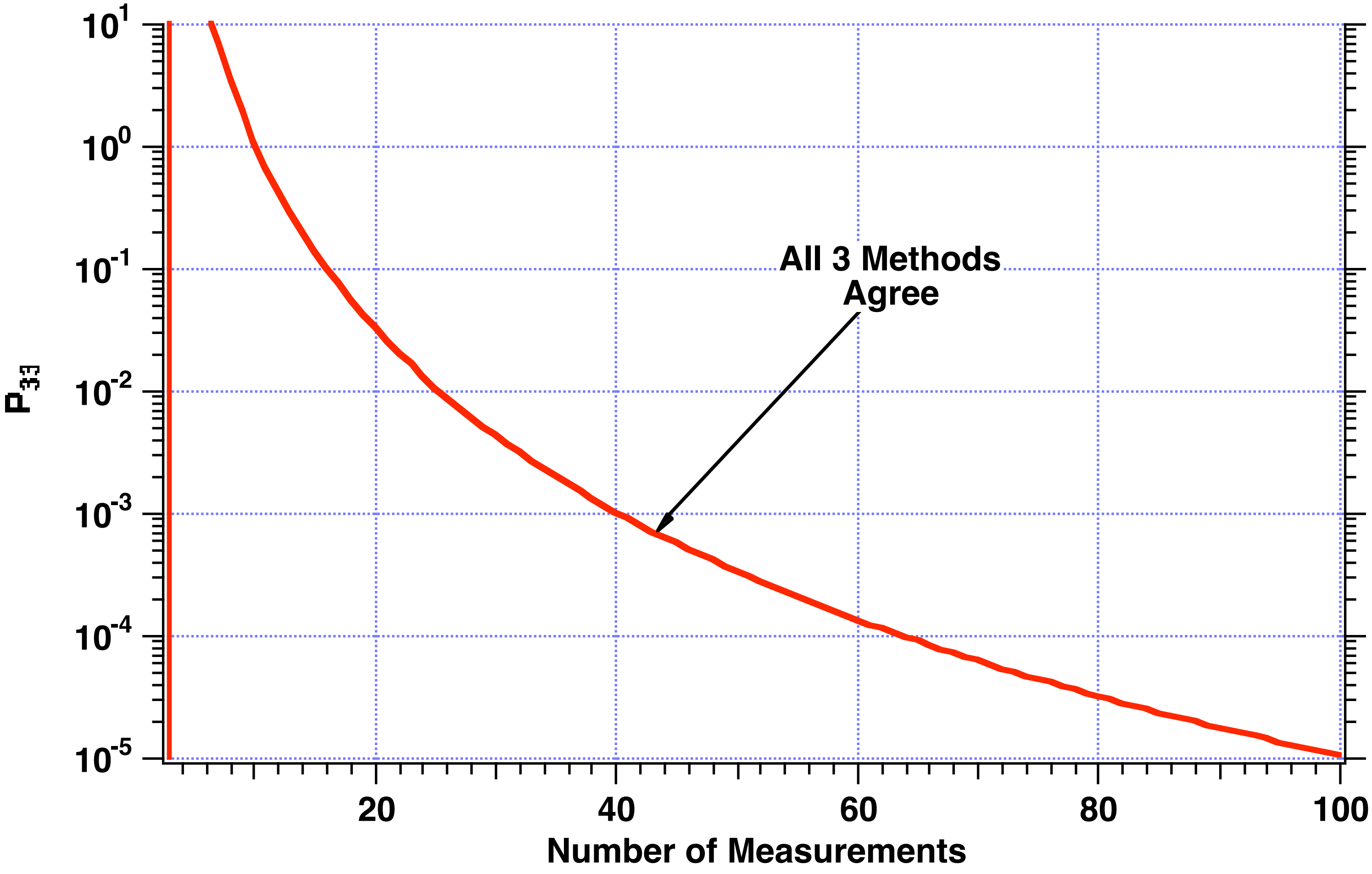
Iterate 3 Methods

All Methods For Finding Best Performance in Three-State System Agree-1



CRLB and Kalman filter with zero process noise and infinite initial covariance matrix results are equivalent

All Methods For Finding Best Performance in Three-State System Agree-2



Observations

- The Cramer-Rao Lower Bound (CRLB) tells us the best a least squares filter can do
    - But so can a recursive least squares filter or the Kalman filter Ricatti equations with zero process noise and infinite initial covariance matrix
- Knowing the best a filter can do does not tell us how to build the filter so that it will work in the real world
- Generally, building a filter with zero process noise is a bad idea because the filter stops paying attention to the measurements
    - Numerous examples have been presented in the course demonstrating how a filter can fall apart with zero process noise

Simple Derivation the CRLB

From Ricatti Equations

$$P = (I - KH)M$$

$$K = MH^T\left(HMH^T + R\right)^{-1}$$

Therefore Substitution Yields

$$P = \left[I - MH^T\left(HMH^T + R\right)^{-1}H\right]M = M - MH^T\left(HMH^T + R\right)^{-1}HM$$

We Want to Prove That

$\underline{P^{-1} = M^{-1} + H^T R^{-1} H}$ **or** $P^{-1} = M^{-1} + H^T R^{-1} H = \left(\Phi P \Phi^T + Q\right)^{-1} + H^T R^{-1} H = \left(\Phi P \Phi^T\right)^{-1} + H^T R^{-1} H$ **if Q=0**

For Preceding Equation to be True

$$I = PP^{-1}$$

$$I = \left[M - MH^T\left(HMH^T + R\right)^{-1}HM\right]\left[M^{-1} + H^T R^{-1} H\right]$$

Multiplying Terms Out and Combining

$$I = I + MH^T\left[R^{-1} - \left(HMH^T + R\right)^{-1}\left(I + HMH^T R^{-1}\right)\right]H$$

But

$$\left(I + HMH^T R^{-1}\right) = \left(R + HMH^T\right)R^{-1}$$

Therefore

$$I = I + MH^T\left[R^{-1} - \left(HMH^T + R\right)^{-1}\left(R + HMH^T\right)R^{-1}\right]H = I + MH^T\left[R^{-1} - IR^{-1}\right]H = I + 0 = I$$