

(Generalized) Positive Functions -

A Convex Invertible Cones Overview

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Outline

- A scalar Convex Invertible Cones (CIC)
- Convex Invertible Cones (CICs) of Matrices
- CICs and the Lyapunov equation
- The Lyapunov order
- CICs of structured matrices.
- CICs of Rational functions.
- Maximal stable CICs
- CICs and KYP Lemma
- Duality impedance - feedback

A scalar Convex Invertible Cone

A. Rantzer,

Int. J. Robust & Non-Lin Cont, 1993

“A weak Kharitonov Thm holds iff the Stability region and its Reciprocal are Convex”.

CICs of Matrices

A set closed under: (i) Positive scaling (ii) Summation
(iii) The inverse of every nonsingular element

- \mathbb{P} set of positive definite matrices is closed under:
Positive scaling
Summation
Inversion } \Rightarrow a Convex Invertible Cone
- $\bar{\mathbb{P}}$ positive semi-definite matrices a CIC,
 \mathbb{P} a non-singular subCIC.

CICs of Matrices (cont.)

- The set of real 2×2 matrices of the form $\begin{pmatrix} + & + \\ - & + \end{pmatrix}$
a convex cone, but not invertible: $\begin{pmatrix} + & + \\ - & + \end{pmatrix}^{-1} = \begin{pmatrix} + & - \\ + & + \end{pmatrix}$
- The set of Hurwitz stable matrices, an invertible cone,
but not convex

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} \quad B = A^T \implies \frac{1}{2}(A + B) = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$$

eigenvalues $\frac{1}{2}(A + B) = -4, +2$ unstable.

The Lyapunov Equation

$\lambda_j(A) + \lambda_k^*(A) \neq 0$ \mathbb{P} positive definite matrices.

$$HA + A^*H \in \mathbb{P}$$

Theorem [M.A. Lyapunov 1892, F.R. Gantmacher 1948]

A is Hurwitz stable $\iff -H \in \mathbb{P}$

Inertia

$$A \in \mathbb{C}^{n \times n} \quad \text{Inertia}(A) = (\nu, \delta, \pi) \quad \nu + \delta + \pi = n$$

ν number of eigenvalues in \mathbb{C}_-

π number of eigenvalues in \mathbb{C}_+

δ number of eigenvalues on $i\mathbb{R}$

$\nu = n \iff$ Hurwitz stable

$\delta = 0 \iff$ Regular inertia

Theorem [A. Ostrowsky & H. Schneider, 1962]

$$\lambda_j(A) + \lambda_k^*(A) \neq 0 \quad H = H^* \text{ nonsingular}$$

$$HA + A^*H \in \mathbb{P} \Rightarrow \text{inertia}(A) = \text{inertia}(H) \quad (\& \text{ regular})$$

CICs & Lyapunov Equation

$$\lambda_j(A) + \lambda_k^*(A) \neq 0 \quad \mathbb{P} \text{ positive definite matrices.}$$

$$HA + A^*H = Q \in \mathbb{P}$$

- $\alpha > 0 \quad H\alpha A + \alpha A^*H = \alpha Q$
- $HA^{-1} + A^{-*}H = A^{-*}Q A^{-1}$
- $HA + A^*H = Q_a \in \mathbb{P} \quad HB + B^*H = Q_b \in \mathbb{P}$

$$H(A + B) + (A + B)^*H = Q_a + Q_b$$

CICs & Lyapunov Equation

$H = H^*$ non-singular \mathbb{P} positive definite matrices

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

$A, B, C \dots \in \mathbb{L}(H) \Rightarrow$

$$(A + 2B^{-1})^{-1} + \frac{1}{2}C + \dots \in \mathbb{L}(H)$$

Rational functions of k non-commuting variables mapping

$$(\mathbb{L}(I))^k \text{ to } \mathbb{L}(I)$$

$$H\mathbb{L}(H) = \mathbb{L}(I)$$

CICs & Lyapunov Equation

\mathbb{P} positive definite matrices

$$\mathbb{L}(I) = \{ A : A + A^* \in \mathbb{P} \}$$

A maximal CIC of matrices with spectrum in \mathbb{C}_+

$$-H \in \mathbb{P}$$

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

“common quadratic Lyapunov function” $x^* H x$

See e.g.

T. Laffey, O. Mason, K.S. Narendra, R. Shorten, H. Šmigoc

CICs & Lyapunov Equation (cont.)

$H = H^*$ non-singular \mathbb{P} positive definite matrices

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

$\mathbb{L}(H)$ a maximal open CIC of matrices

sharing the same inertia (as H)

Maximality

Any B s.t. $HB + B^*H$ has a negative eigenvalue \Rightarrow

$\exists A$ in $\mathbb{L}(H)$ s.t. $A + B$ has an eigenvalue on $i\mathbb{R}$

CICs & Lyapunov Equation (cont.)

$H = H^*$ non-singular

\mathbb{P} positive definite matrices

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

Theorem [T. Ando, 2001, 2004]

A maximal open Convex Invertible Cone of matrices

sharing the same inertia (along with additional conditions)

$$\Rightarrow \mathbb{L}(H).$$

Conclusion: $\mathbb{L}(H)$ a “prototype” of all maximal open

non-singular CICs

CICs & Lyapunov Equation (cont.)

Given $A, B \in \mathbb{C}^{n \times n}$

- (i) $\exists H = H^*$ non-singular s.t. $A, B \in \mathbb{L}(H)$
- (ii) All matrices in $\text{CIC}(A, B)$ are non-singular.
- (iii) All matrices in $\text{conv}(A, A^{-1}, B, B^{-1})$ have regular inertia.
- (iv) A, B have the same regular inertia and both $\text{conv}(A, B)$ and $\text{conv}(A, B^{-1})$ are non-singular.

Then, (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)

If $A, B \in \mathbb{R}^{2 \times 2}$ then (iv) \Rightarrow (i)

CICs & Lyapunov Equation (cont.)

$$A \in \mathbb{R}^{n \times n}$$

$$\lambda_j(A) + \lambda_k^*(A) \neq 0$$

(stronger than regular inertia)

$$\mathbb{H}(A) = \{H = H^* \text{ real} : HA + A^*H \in \mathbb{P}\}$$

Theorem [R. Loewy, 1976] $A, B \in \mathbb{R}^{n \times n}$

$$\mathbb{H}(A) = \mathbb{H}(B) \iff \text{CIC}(A) = \text{CIC}(B)$$

A Generalization of CICs & Lyap. Eq.

$H = H^*$ nonsingular

$$\mathbb{L}(H) = \{ A \in \mathbb{C}^{n \times n} : HA + A^*H \in \mathbb{P} \}$$

$A, B \in \mathbb{L}(H) \Rightarrow$

$$((A + 3B + ir_1I)^{-1} + ir_2I + \frac{1}{2}A)^{-1} \dots \in \mathbb{L}(H)$$

$$\forall r_1, r_2 \in \mathbb{R}$$

For results on a pair of matrices see

R. Loewy 1976 $\mathbb{H}(A) = \mathbb{H}(B) \subset \mathbb{C}^{n \times n}$

T. Laffey H. Smigoc 2007 $\mathbb{H}(A) \cap \mathbb{H}(B) \neq \emptyset \subset \mathbb{C}^{2 \times 2}$

CICs of Structured Matrices

[Cohen & L. 1997]

\mathbb{A} a CIC in $\mathbb{C}^{n \times n}$

\mathbb{B} a CIC in $\mathbb{C}^{m \times m}$

Fix a non-singular $T \in \mathbb{C}^{(n+m) \times (n+m)}$

$$\mathbb{M} = \left\{ T \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} T^{-1} : A \in \mathbb{A}, B \in \mathbb{B} \right\}$$

$\Rightarrow \mathbb{M}$ a CIC in $\mathbb{C}^{(n+m) \times (n+m)}$

Take T structured [L. 1999].

CICs of Hamiltonians

Example, the Sylvester equation (Lyapunov, Riccati ...)

$A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{m \times m}$, $C \in \mathbb{C}^{m \times n}$ data

$XA + BX = C$ $X \in \mathbb{C}^{m \times n}$ solution

M associated Hamiltonian

$$M = \begin{pmatrix} A & \mathbf{0} \\ C & -B \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix} \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & -B \end{pmatrix} \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix}^{-1}$$

Fix X and consider $XA_j + B_jX = C_j$ $j = 1, 2 \dots$

Equivalently,

$$M_j = \begin{pmatrix} A_j & \mathbf{0} \\ C_j & -B_j \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix} \begin{pmatrix} A_j & \mathbf{0} \\ \mathbf{0} & -B_j \end{pmatrix} \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix}^{-1} \quad j = 1, 2 \dots$$

CICs of Hamiltonians (cont.)

$$M_j = \begin{pmatrix} A_j & \mathbf{0} \\ C_j & -B_j \end{pmatrix} = \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix} \begin{pmatrix} A_j & \mathbf{0} \\ \mathbf{0} & -B_j \end{pmatrix} \begin{pmatrix} I_n & \mathbf{0} \\ X & I_m \end{pmatrix}^{-1} \quad j = 1, 2 \dots$$

Assume

$$A_j \in \text{CIC } \mathbb{A} \subset \mathbb{C}^{n \times n} \qquad \qquad B_j \in \text{CIC } \mathbb{B} \subset \mathbb{C}^{m \times m}$$
$$\Rightarrow M_j \in \text{CIC } \mathbb{M} \subset \mathbb{C}^{(n+m) \times (n+m)}$$

For simplicity \mathbb{A} and \mathbb{B} Hurwitz stable \Rightarrow

$\begin{pmatrix} -I_n & \mathbf{0} \\ -2X & I_m \end{pmatrix}$ is the only involution in the CIC \mathbb{M}

\mathbb{M} a CIC of Hamiltonians associated with Sylvester equations

with a common solution X

CICs of Hamiltonians (cont.)

For the Lyapunov equation, take $B = A^*$

Fix $X_a = X_a^*$ $X_b = X_b^*$ s.t. $(X_a + X_b)$ non-singular

$$M_j = \left(\begin{smallmatrix} I & I \\ X_a & -X_b \end{smallmatrix} \right) \left(\begin{smallmatrix} A_j & 0 \\ 0 & -B_j \end{smallmatrix} \right) \left(\begin{smallmatrix} I & I \\ X_a & -X_b \end{smallmatrix} \right)^{-1} \quad j = 1, 2 \dots$$

Assume

$$\begin{aligned} A_j \in \text{CIC } \mathbb{A} \subset \mathbb{C}^{n \times n} & \qquad \qquad \qquad B_j \in \text{CIC } \mathbb{B} \subset \mathbb{C}^{n \times n} \\ & \Rightarrow M_j \in \text{CIC } \mathbb{M} \subset \mathbb{C}^{2n \times 2n} \end{aligned}$$

\mathbb{M} a CIC of Hamiltonians associated with Riccati equations

with a common pair of solution X_a X_b

Scalar Rational Positive Real Functions

\mathbb{C}_+ ($\overline{\mathbb{C}_+}$) open (closed) right half plane.

\mathcal{PR} real rational functions analytically mapping $\mathbb{C}_+ \rightarrow \overline{\mathbb{C}_+}$

$$f \in \mathcal{PR} \Rightarrow \operatorname{Re} f(s) \geq 0 \quad s \in \mathbb{C}_+$$

The Nyquist plot is in $\overline{\mathbb{C}_+}$ (infinite gain margin)

Scalar \mathcal{PR} (cont.)

$$f \in \mathcal{PR} \Rightarrow \operatorname{Re} f(s) \geq 0 \quad s \in \mathbb{C}_+$$

Driving point immittance of R-L-C electrical networks

W. Cauer 1926

O. Brune 1931

Linear passive/dissipative networks:

Belevitch 1968, Wohlers 1969

Lurie problem, hyper/absolute stability:

K.S. Narendra & J.H. Taylor 1973, V. Popov 1973

CIC - Electrical Networks Interpretation

The set \mathcal{PR} is a Convex Invertible Cone CIC:

closed under (i) positive scaling (ii) addition (iii) inversion.

Electrical networks CIC interpretation of \mathcal{PR}

Scaling - transformer ratio

Summation - series connection of impedances

Inversion - impedance / admittance

CICs of Stable Functions

Observation: [Cohen & L, 2007]

$\pm \mathcal{PR}$ are maximal CICs

of real rational functions analytic in \mathbb{C}_+

(i.e. stable minimum phase)

Positive Real Odd functions \mathcal{PRO} - a subCIC of \mathcal{PR}

$$f \in \mathcal{PRO} \iff f \in \mathcal{PR} \text{ & } f(-s) = -f(s)$$

Driving point immittance of L-C circuits (Lossless, Foster).

Maximal Stable CICs

Constant matrices

$$\mathbb{P} \text{ positive definite matrices} \quad -H \in \mathbb{P}$$

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

A maximal open CIC of Hurwitz stable matrices.

Rational functions

$$F(s) \in \mathcal{P} \quad F(s) + F(s)^*_{|s \in \mathbb{C}_+} \in \overline{\mathbb{P}} \quad \left(F(s)_{|s \in \mathbb{C}_+} \in \overline{\mathbb{L}}(I) \right)$$

A maximal CIC of stable minimum phase rational functions

Nevanlinna-Pick & a CIC

$$A, B \in \mathbb{R}^{n \times n} \quad \lambda_j(A) + \lambda_k^*(A) \neq 0 \quad \lambda_j(B) + \lambda_k^*(B) \neq 0$$

$$\mathbb{H}(A) = \{H = H^* \text{ real} : HA + A^*H \in \mathbb{P}\}$$

Observation TFAE

- $B \in \text{CIC}(A)$
- $B = f(A) \quad f \in \mathcal{PRO}$
- $\mathbb{H}(A) \subset \mathbb{H}(B)$

Conclusion: CIC induces a Lyapunov order $B \geq A$

Lyapunov Order

$$\mathbb{H}(A) \subset \mathbb{H}(B)$$

Every quadratic Lyapunov function for A is applicable to B

In a sense $\dot{x} = Bx$ is "more stable" than $\dot{x} = Ax$.

In principle, one can have

$\dot{x} = f_b(x)$ "more stable" than $\dot{x} = f_a(x)$

A Lyapunov order for non-linear systems

Example: $f_a(x) = -x|x|$ $f_b(x) = f_a(x) - \sin(x)$

Rational Gen. Pos. Functions \mathcal{GP}

Rational (scalar) generalized positive functions \mathcal{GP}

$$f \in \mathcal{P} \implies \operatorname{Re} f(s) \geq 0 \quad s \in \mathbb{C}_+$$

$$f \in \mathcal{GP} \implies \operatorname{Re} f(s) \geq 0 \quad \text{a.e. } s \in i\mathbb{R}$$

Why should one be interested in \mathcal{GP} functions?

A Characterization of Positive Functions

\mathcal{P} functions analytically mapping $\mathbb{C}_+ \rightarrow \overline{\mathbb{C}_+}$

\mathcal{GP} functions a.e. analytically mapping $i\mathbb{R} \rightarrow \overline{\mathbb{C}_+}$

Theorem [Wohlers, 1969]

A rational $f \in \mathcal{P} \iff$ both $f(s)$ and $1/f(s)$ are:

(i) Analytic in \mathbb{C}_+

(ii) Generalized Positive \mathcal{GP}

(iii) On $i\mathbb{R}$ all poles are simple and have positive residue

Generalized Positive Functions

$$\mathcal{P} := \{ F(s) : F(s)_{|s \in \mathbb{C}_+} \in \overline{\mathbf{L}}(I) \}$$

$$\mathcal{GP} := \{ F(s) : F(s)_{|s \in i\mathbb{R}} \in \overline{\mathbf{L}}(I) \}$$

$$G^\#(s) := (G(-s^*))^* \quad G^\#(s)_{|s \in i\mathbb{R}} = (G(s)_{|s \in i\mathbb{R}})^*$$

$$F \in \mathcal{GP} \iff F(s) = G(s)P(s)G^\#(s) \quad P \in \mathcal{P}$$

G and G^{-1} analytic in \mathbb{C}_+

CICs of Rational Functions

$$\mathcal{P} := \{ F(s) : F(s)_{|s \in \mathbb{C}_+} \in \overline{\mathbf{L}}(I) \}$$

$$\mathcal{GP} := \{ F(s) : F(s)_{|s \in i\mathbb{R}} \in \overline{\mathbf{L}}(I) \}$$

$$\mathcal{GPE} := \{ F(s) : F(s)_{|s \in i\mathbb{R}} \in \overline{\mathbb{P}} \}$$

$$\mathcal{O}dd := \{ F(s) : F^\#(s) = -F(s) \}$$

$$\mathcal{PO} := \mathcal{P} \cap \mathcal{O}dd$$

Rational functions of k non-commuting variables mapping

$$(\mathbb{L}(I))^k \text{ to } \mathbb{L}(I)$$

(Generalized) K-Y-P Lemma

$F(s)$ a (matrix valued) rational function

$$F(s) \in \mathcal{GP} \quad F(s)_{|s \in i\mathbb{R}} \in \overline{\mathbf{L}}(I) \quad \text{a.e.} \quad \exists \lim_{s \rightarrow \infty} F(s)$$

State space realization $F(s) = C(sI - A)^{-1}B + D$

B. Dickinson, Ph. Delsarte, Y. Genin & Y. Kamp, 1985

$$F \in \mathcal{GP} \iff \exists \quad H = H^* \text{ nonsingular s.t.}$$

$$\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) + \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \in \overline{\mathbb{P}}$$

CICs & Linear Matrix Inequalities

$H = H^*$ non-singular

\mathbb{P} positive definite matrices

$$\mathbb{L}(H) = \{ A : HA + A^*H \in \mathbb{P} \}$$

maximal open non-singular CIC

$$\begin{aligned} \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) + \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \\ = \begin{pmatrix} HA + A^*H & HB + C^* \\ B^*H + C & D + D^* \end{pmatrix} \in \overline{\mathbb{P}} \end{aligned}$$

A typical LMI

(Generalized) K-Y-P Lemma (cont.)

$$F \in \mathcal{GP} \iff \exists \quad H = H^* \text{ nonsingular s.t.}$$

$$\begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) + \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^* \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \in \bar{\mathbb{P}}$$

- Minimality of the realization
- $\mathbb{L} \left(\begin{smallmatrix} H & 0 \\ 0 & I \end{smallmatrix} \right)$ a maximal state space CICs e.g.

$$0.3 \left(\begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right) + \left(\begin{array}{cc} A_2 & B_2 \\ C_2 & D_2 \end{array} \right)^{-1} \dots$$

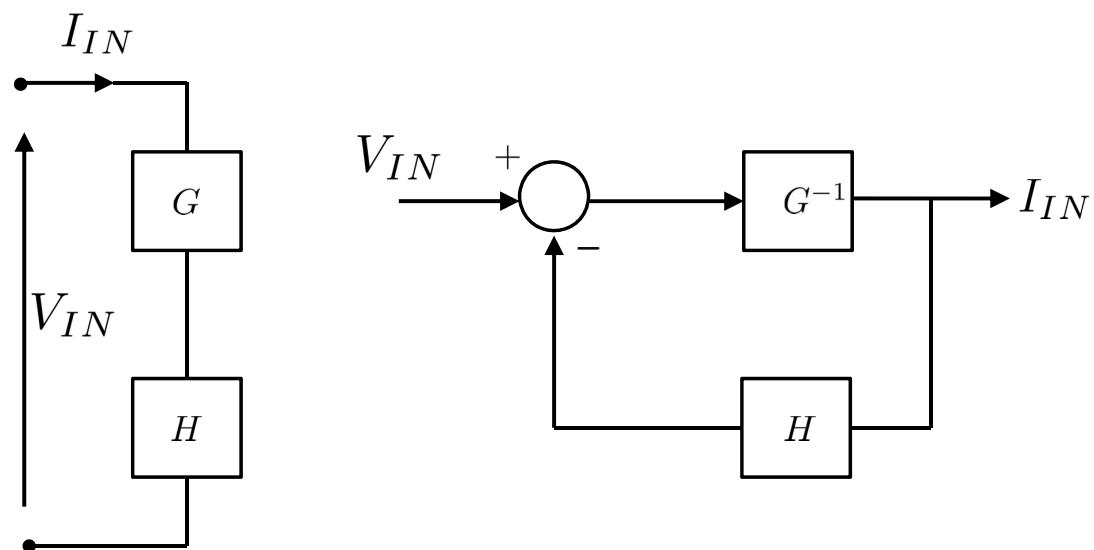
preserving the McMillan degree

- $F \in \mathcal{P} \iff -H \in \mathbf{P}$

Duality Imdepence - Feedback

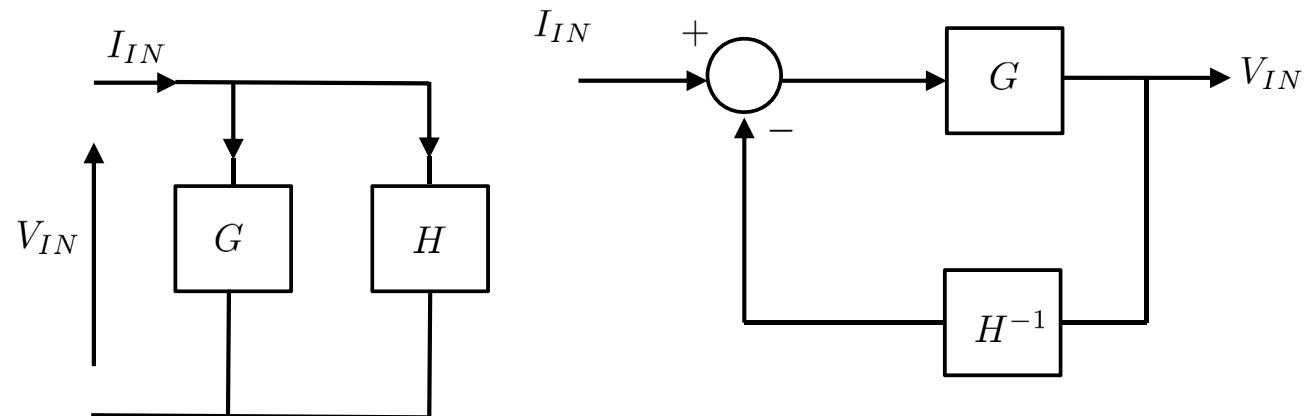
$$I_{IN} = (G + H)^{-1} V_{IN}$$

$$\det(G), \det(H), \det(H + G) \not\equiv 0$$



Duality Admittance - Feedback

$$V_{IN} = (H^{-1} + G^{-1})^{-1} I_{IN}$$
$$\det(G), \det(H), \det(H + G) \neq 0$$



CICs and \mathcal{GP} - References

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THANKS FOR YOUR ATTENTION