Cooperative Optimal Missile Guidance Laws

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Introduction

Guidance is the process of modifying the trajectory of a vehicle in motion in order to reach a pre-specified target. The target is, in the most general sense, a set of states (position, velocity) either fixed in time or time varying.

Classical guidance laws

- Attitude/velocity pursuit (Pure Pursuit)
- Command-to-Line-of Sight / Beam Rider
- Proportional Navigation (Parallel Navigation)
- Optimal control based guidance
- ...

All these guidance laws are **one-on-one**

Introduction

Motivation for cooperative guidance

- Saturate the target’s defenses (e.g. using a salvo attack)
- Limit the target’s evasive possibilities (e.g. by controlling the relative geometry)
- Lure the target to a trap

Potentially two possible modes of cooperative guidance:

- Implicit cooperation - some coordination parameter between vehicles (e.g. intercept target from pre-specified angels or at pre-specified time)
- Explicit cooperation - optimizing team performance criteria (e.g. enforcing relative geometry in-between missile team or simultaneous interception)
- Explicit cooperation can yield better performance (e.g. by considering intrinsic relationships between team members)

Introduction (contd.)

Introduction - related literature

Intercept angle guidance

- PN based laws (Kim et al., Ratnoo & Ghose, etc.)
- Optimal guidance laws (Ryoo et al., Idan et al., etc.)
- Maneuvering target (Shima)
- Differential Games laws (Shaferman & Shima)

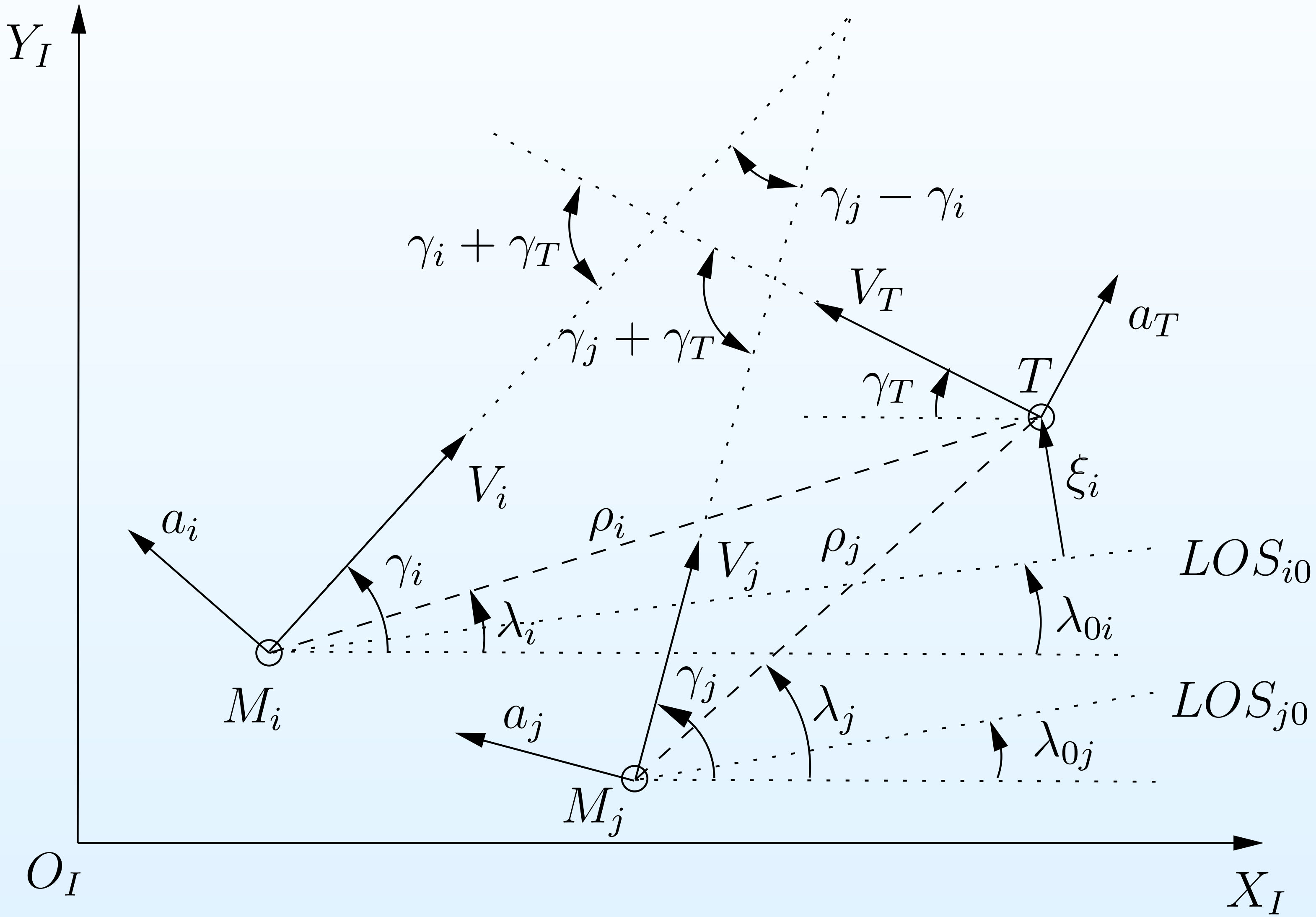
Cooperative guidance laws

- Differential games (Hagedorn & Breakwell, etc.)
- Simultaneous arrival (Lee et al., Meyer et al.)
- ...

Outline

- $\surd$ Introduction
- **Mathematical Model**
- Optimization Problem
- Optimal Cooperative Law
- Performance Analysis
- Summary & Conclusions

Mathematical Model - planar engagement geometry



Mathematical Model - non linear EOM

$$\dot{\rho}_i = V_{\rho i} \tag{1a}$$

$$\dot{\lambda}_i = V_{\lambda i}/\rho_i \tag{1b}$$

$$\dot{\gamma}_T = a_T/V_T \tag{1c}$$

$$\dot{a}_T = (u_T - a_T)/\tau_T \tag{1d}$$

where

$$V_{\rho i} = -\left[V_T \cos(\gamma_T + \lambda_i) + V_i \cos(\gamma_i - \lambda_i)\right] \tag{2a}$$

$$V_{\lambda i} = V_T \sin(\gamma_T + \lambda_i) - V_i \sin(\gamma_i - \lambda_i) \tag{2b}$$

Once collision triangle i is reached and maintained then $V_{\rho i}$ is constant and interception time can be assumed fixed:

$$t_{fi} = -\rho_{i0}/V_{\rho i} \tag{3}$$

Mathematical Model - non linear EOM (contd.)

Each missile's dynamics and acceleration (dim($\boldsymbol{y}_i) = l$):

$$\dot{\boldsymbol{y}}_i = \mathbf{A}_i \boldsymbol{y}_i + \mathbf{B}_i u_i \tag{4a}$$

$$\dot{\gamma}_i = a_i / V_i \tag{4b}$$

$$a_i = \mathbf{C}_i \boldsymbol{y}_i + D_i u_i \tag{4c}$$

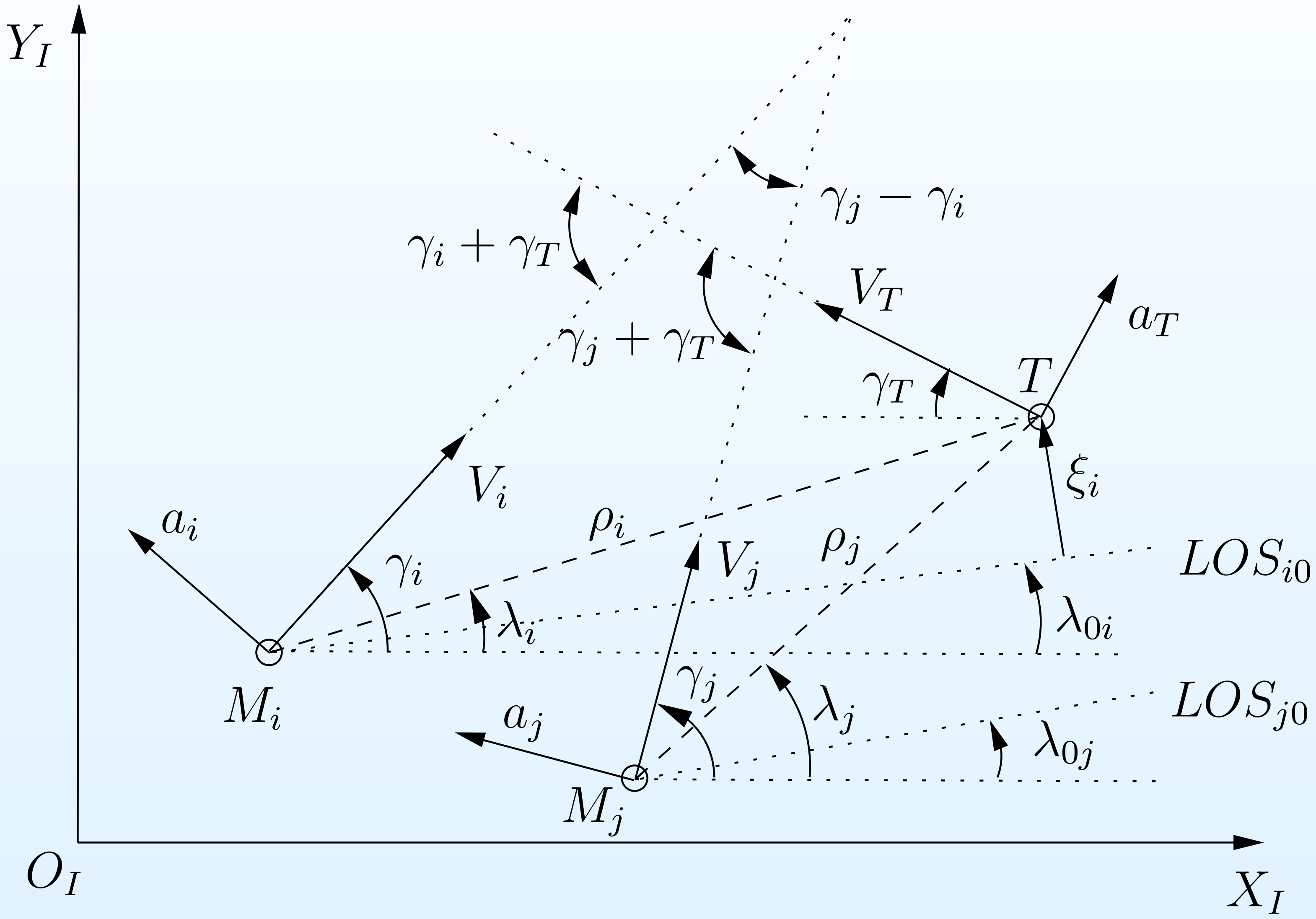
The missiles' internal states, accelerations, and controls are:

$$\boldsymbol{x}_m = \begin{bmatrix} \boldsymbol{y}_1^T & \boldsymbol{y}_2^T & \dots & \boldsymbol{y}_n^T \end{bmatrix}^T \tag{5}$$

$$\boldsymbol{a}_m = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}^T \tag{6}$$

$$\boldsymbol{u}_m = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}^T \tag{7}$$

Mathematical Model - linearized EOM



Mathematical Model - linearized EOM

The state vector of the cooperative linearized problem is

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1^T & \boldsymbol{x}_2^T & \boldsymbol{x}_\gamma^T & \boldsymbol{x}_m^T & a_T \end{bmatrix}^T \tag{8}$$

where $\boldsymbol{x}_1$ is a vector of the n separations between the n missiles and the target, relative to LOS_{i0}

$$\boldsymbol{x}_1 = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_n \end{bmatrix}^T \tag{9}$$

and $\boldsymbol{x}_2$ is its derivative. $\boldsymbol{x}_\gamma$ is a vector of the respective flight path angles

$$\boldsymbol{x}_\gamma = \begin{bmatrix} \gamma_T + \gamma_1 & \gamma_T + \gamma_2 & \ldots & \gamma_T + \gamma_n \end{bmatrix}^T \tag{10}$$

dim $(\boldsymbol{x}) = n(3 + l) + 1$

Mathematical Model - linearized EOM (contd.)

The equations of motion are

$$\dot{\boldsymbol{x}} = \begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 = \mathbf{E}_T a_T - \mathbf{E}_m \boldsymbol{a}_m \\ \dot{\boldsymbol{x}}_\gamma = \mathbf{F}_T a_T + \mathbf{F}_m \boldsymbol{a}_m \\ \dot{\boldsymbol{x}}_m = \mathbf{A}_m \boldsymbol{x}_m + \mathbf{B}_m \boldsymbol{u}_m \\ \dot{a}_T = (u_T - a_T)/\tau_T \end{cases} \tag{11}$$

Matrix form of the equation set

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}_m + \mathbf{C}u_T \tag{12}$$

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Optimization Problem - explicit cooperation

Ordered interception times

$$t_{fn} \geq \ldots \geq t_{f2} \geq t_{f1} \tag{13}$$

Cost function (explicit)

$$\begin{aligned} J = \frac{\alpha_1}{2}\xi_1^2(t_{f1}) + \ldots + \frac{\alpha_n}{2}\xi_n^2(t_{fn}) + \frac{\beta_1}{2}[\boldsymbol{x}_{\gamma 1}(t_{f1}) - \boldsymbol{x}_{\gamma 2}(t_{f2}) - \Delta_{c1}]^2 \\ + \ldots + \frac{\beta_{n-1}}{2}[\boldsymbol{x}_{\gamma(n-1)}(t_{f(n-1)}) - \boldsymbol{x}_{\gamma n}(t_{fn}) - \Delta_{c(n-1)}]^2 \\ + \frac{1}{2}\int_0^{t_{f1}} \eta_1^2 u_1^2 dt + \ldots + \frac{1}{2}\int_0^{t_{fn}} \eta_n^2 u_n^2 dt \end{aligned} \tag{14}$$

where the weights α_i, β_i, & η_i are non-negative. $\boldsymbol{x}_{\gamma i} = \gamma_T + \gamma_i$

Optimization Problem - implicit cooperation

Example for implicit cost function

$$J = \frac{1}{2}\sum_{i=1}^{n} J_i \tag{15}$$

where

$$J_i = \alpha_i \xi_i^2(t_{fi}) + \beta_i[\gamma_i(t_{fi}) - \gamma_T(t_{fi}) - \Delta_{ci}]^2 + \int_0^{t_{fi}} \eta_i^2 u_i^2 dt \tag{16}$$

In such an engagement each missile will be guided using the appropriate 1-on-1 guidance law, which for this case is that of [Shaferman & Shima, AIAA Journal of Guidance, Control, and Dynamics, 2008] .

Optimization Problem - order reduction

One sided order reduction

$$\mathbf{Z}(t) \;= \mathbf{D}\mathbf{\Phi}(t_f, t)\boldsymbol{x}(t) + \mathbf{D}\int_t^{t_f} \mathbf{\Phi}(t_f, \tau)\mathbf{C}u_T d\tau \tag{17}$$

The time derivative of the new state $\mathbf{Z}(t)$ is

$$\dot{\mathbf{Z}} = \mathbf{D}[\dot{\mathbf{\Phi}}(t_f, t)\boldsymbol{x} + \mathbf{\Phi}(t_f, t)\dot{\boldsymbol{x}}] - \mathbf{D}\mathbf{\Phi}(t_f, t)\mathbf{C}u_T = \mathbf{D}\mathbf{\Phi}(t_f, t)\mathbf{B}u_m \tag{18}$$

Zero effort miss of missile 1 (Z_1) is obtained for

$$\mathbf{D} = \mathbf{D}_\xi = \begin{bmatrix} 1 & 0 & 0 & [0] & 0 \end{bmatrix} \tag{19}$$

Zero effort angle (ZEA) of missile 1 (Z_{n+1}) is obtained for

$$\mathbf{D} = \mathbf{D}_\gamma = \begin{bmatrix} 0 & 0 & 1 & [0] & 0 \end{bmatrix} \tag{20}$$

Optimization Problem - order reduction (contd.)

$$J = \frac{\alpha_1}{2} Z_1^2(t_{f1}) + ... + \frac{\alpha_n}{2} Z_n^2(t_{fn}) + \frac{\beta_1}{2}[Z_{n+1}(t_{f1}) - Z_{n+2}(t_{f2}) - \Delta_{c1}]^2 \quad (21)$$
$$+ ... + \frac{\beta_{n-1}}{2}[Z_{2n-1}(t_{f(n-1)}) - Z_{2n}(t_{fn}) - \Delta_{c(n-1)}]^2$$
$$+ \frac{1}{2}\int_0^{t_{f1}} \eta_1^2 u_1^2 dt + ... + \frac{1}{2}\int_0^{t_{fn}} \eta_n^2 u_n^2 dt$$

Missile i ceases to exist for $t > t_{fi}$. Therefore $u_i = 0$ for $t > t_{fi}$

$$J = \frac{\alpha_1}{2} Z_1^2(t_{fn}) + ... + \frac{\alpha_n}{2} Z_n^2(t_{fn}) + \frac{\beta_1}{2}[Z_{n+1}(t_{fn}) - Z_{n+2}(t_{fn}) - \Delta_{c1}]^2 \quad (22)$$
$$+ ... + \frac{\beta_{n-1}}{2}[Z_{2n-1}(t_{fn}) - Z_{2n}(t_{fn}) - \Delta_{c(n-1)}]^2$$
$$+ \frac{1}{2}\int_0^{t_{fn}} \eta_1^2 u_1^2 dt + ... + \frac{1}{2}\int_0^{t_{fn}} \eta_n^2 u_n^2 dt$$

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Optimal Cooperative Law - Derivation

The Hamiltonian of the problem is

$$H = \frac{1}{2}(\eta_1^2 u_1^2 + \eta_2^2 u_2^2 + ... + \eta_n^2 u_n^2) + \dot{Z}_1 \lambda_{z_1} + \dot{Z}_2 \lambda_{z_2} + ... + \dot{Z}_{2n} \lambda_{z_{2n}} \quad (23)$$

Time derivative of zero effort variables is state independent, simplifying considerably the adjoint equations. Resulting with

$$\lambda_{z_i}(t) = \alpha_i Z_i(t_{f_n}), \quad i \in \{1, ..., n\} \quad (24a)$$

$$\frac{1}{\beta_i} \sum_{j=1}^{i} \lambda_{z_{n+j}} = \Delta Z_{n+i}(t_{f_n}) \quad , \quad i \in \{1, ..., n-1\} \; , \; \sum_{j=1}^{n} \lambda_{z_{n+j}} = 0 \quad (24b)$$

where the relative i & $i+1$ angular error at intercept is

$$\Delta Z_{n+i}(t_{f_n}) = Z_{n+i}(t_{fn}) - Z_{n+i+1}(t_{f_n}) - \Delta_{ci} \quad (25)$$

Optimal Cooperative Law - Derivation

The reduced order dynamics associated with the i-th missile is (Eq. (18))

$$\dot{\mathbf{Z}}^{(i)} = \begin{cases} \dot{Z}_i = \mathbf{D}_\xi \mathbf{\Phi}^{(i)}(t_{go_i})\mathbf{B}^{(i)}u_i = \varphi_\xi^{(i)}(t_{go_i})u_i \\ \dot{Z}_{n+i} = \mathbf{D}_\gamma \mathbf{\Phi}^{(i)}(t_{go_i})\mathbf{B}^{(i)}u_i = \varphi_\gamma^{(i)}(t_{go_i})u_i \end{cases} \quad (26)$$

Therefore, the optimal controllers for the missiles, which satisfy $u_i^* = \arg\min_{u_i} H, \quad i \in \{1, 2, ..., n\}$ are

$$u_i^*(t) = -\frac{1}{\eta_i^2}\sum_{j=1}^{2n} \lambda_{z_j}(t)\frac{\partial \dot{Z}_j}{\partial u_i} = -\frac{1}{\eta_i^2}\left[\lambda_{z_i}(t)\varphi_\xi^{(i)}(t_{go_i}) + \lambda_{z_{n+i}}(t)\varphi_\gamma^{(i)}(t_{go_i})\right] \quad (27)$$

Optimal Cooperative Law - Derivation

Substituting Eq. (27) into Eq. (26) and integrating from t to t_{f_i} yields the following $2n$ coupled algebraic equations for $i \in \{1, ..., n\}$

$$Z_i(t_{f_n}) = Z_i(t_{f_i}) = Z_i(t) - \lambda_{z_i} \chi_{\xi\xi}^{(i)}(t_{go_i}) - \lambda_{z_{n+i}} \chi_{\xi\gamma}^{(i)}(t_{go_i}) \tag{28a}$$

$$Z_{n+i}(t_{f_n}) = Z_{n+i}(t_{f_i}) = Z_{n+i}(t) - \lambda_{z_i} \chi_{\xi\gamma}^{(i)}(t_{go_i}) - \lambda_{z_{n+i}} \chi_{\gamma\gamma}^{(i)}(t_{go_i}) \tag{28b}$$

where

$$\chi_{jl}^{(i)}(t_{go_i}) = \frac{1}{\eta_i^2} \int_0^{t_{go_i}} \varphi_j^{(i)}(t_{go}) \varphi_l^{(i)}(t_{go}) dt_{go} \quad , \quad j, l \in \{\xi, \gamma\} \tag{29}$$

There are therefore $4n$ coupled linear equations with $4n$ unknowns $Z_i(t_{f_n}), Z_{n+i}(t_{f_n}), \lambda_i, \lambda_{n+i}, \quad i \in \{1, ..., n\}$.

Optimal Cooperative Law - Derivation

Closed form solution obtained for **any team size** n

$$\lambda_{z_{n+1}} = \frac{1}{\Psi_1 - \Upsilon^{(1)}(t_{go_1})}\left[\Gamma^{(1)}(t) + \sum_{j=2}^{n-1}\Gamma^{(j)}(t)\prod_{p=2}^{j}\frac{-\Upsilon^{(j)}(t_{go_j})}{\Psi_j - \Upsilon^{(j)}(t_{go_j})}\right] \tag{30}$$

where

$$\Psi_j(t_{go_{j+1}}, t_{go_{j+2}}, ..., t_{go_n}) = \begin{cases} \frac{\left(\Psi_{j+1}-\Upsilon^{(j+1)}\right)-\Psi_{j+1}\Upsilon^{(j+1)}}{\beta_j\left(\Psi_{j+1}-\Upsilon^{(j+1)}\right)} & , j \in \{1, ..., n-2\} \\ \frac{1-\beta_{n-1}\Upsilon^{(n)}}{\beta_{n-1}} & , j \in \{n-1\} \end{cases} \tag{31}$$

The rest of the co-states $\lambda_{z_{n+i}}$ solved by an iterative process

$$\lambda_{z_{n+i+1}} = \frac{1}{\Upsilon^{(i+1)}(t_{go_{i+1}})}\left[-\frac{1}{\beta_i}\sum_{j=1}^{i}\lambda_{z_{n+j}} + \Upsilon^{(i)}(t_{go_i})\lambda_{z_{n+i}} + \Gamma^{(i)}(t)\right] \tag{32}$$

Optimal Cooperative Law - Derivation

Substituting and rearranging the optimal controller of the i-th missile is

$$u_i^*(t) = \sum_{j=1}^{n} \frac{N_{Z_j}^{u_i}}{t_{go_i}^2} Z_j(t) + \sum_{j=1}^{n-1} N_{\Delta Z_{n+j}}^{u_i} \frac{V_i}{t_{go_i}} \Delta Z_{n+j}(t), \;\; t \in [0, t_{f_i}], i \in \{1, ..., n\} \tag{33}$$

It is linear with the ZEM variables and the relative ZEAE variables between consecutive missiles

When the relative geometry between the missiles is not imposed ($\beta_i \to 0$), the solution is reduced to n decoupled guidance laws

$$u_i^*(t) = \frac{-\alpha_i \varphi_\xi^{(i)}(t_{go_i}) Z_i(t)}{\eta_i^2 \left[1 + \alpha_i \chi_{\xi\xi}^{(i)}(t_{go_i})\right]} \quad , \quad t \in [0, t_{f_i}] \quad , \quad i \in \{1, ..., n\} \tag{34}$$

Optimal Cooperative Law - Properties

Imposing perfect intercept and intercept angle ($\alpha_i, \beta_i \to \infty,\ \forall i$)

$$N_{Z_k}^{u_i} = \begin{cases} \dfrac{\left[\chi_{\xi\gamma}^{(i)}\varphi_{\xi}^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_{\gamma}^{(i)}\right] t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty} \left[\sum_{j=1}^{n} \dfrac{\Lambda^{(i)}}{\Upsilon^{(j)}} - \dfrac{\Lambda^{(i)}}{\Upsilon^{(i)}}\right] - \dfrac{\varphi_{\xi}^{(i)} t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)}} & ,\ k = i \\ \dfrac{-\left[\chi_{\xi\gamma}^{(i)}\varphi_{\xi}^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_{\gamma}^{(i)}\right] \Lambda^{(k)} t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty \Upsilon^{(k)}} & ,\ k \neq i \end{cases}$$

$$N_{\Delta Z_{n+k}}^{u_i} = \begin{cases} \dfrac{\left[\chi_{\xi\gamma}^{(i)}\varphi_{\xi}^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_{\gamma}^{(i)}\right] t_{go_i}}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty V_i} \sum_{j=1}^{k} \dfrac{1}{\Upsilon^{(j)}} & ,\quad k < i \\ \dfrac{-\left[\chi_{\xi\gamma}^{(i)}\varphi_{\xi}^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_{\gamma}^{(i)}\right] t_{go_i}}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty V_i} \sum_{j=k+1}^{n} \dfrac{1}{\Upsilon^{(j)}} & ,\quad k \geq i \end{cases}$$

Optimal Cooperative Law - Properties

Imposing perfect intercept and intercept angle ($\alpha_i, \beta_i \to \infty,\ \forall i$)

$$N_{Z_k}^{u_i} = \begin{cases} \dfrac{\left[\chi_{\xi\gamma}^{(i)}\varphi_\xi^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_\gamma^{(i)}\right] t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty} \left[\displaystyle\sum_{j=1}^{n} \frac{\Lambda^{(i)}}{\Upsilon^{(j)}} - \frac{\Lambda^{(i)}}{\Upsilon^{(i)}}\right] - \dfrac{\varphi_\xi^{(i)} t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)}} & , \ k = i \\[2ex] \dfrac{-\left[\chi_{\xi\gamma}^{(i)}\varphi_\xi^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_\gamma^{(i)}\right] \Lambda^{(k)} t_{go_i}^2}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty \Upsilon^{(k)}} & , \ k \neq i \end{cases}$$

$$N_{\Delta Z_{n+k}}^{u_i} = \begin{cases} \dfrac{\left[\chi_{\xi\gamma}^{(i)}\varphi_\xi^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_\gamma^{(i)}\right] t_{go_i}}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty V_i} \displaystyle\sum_{j=1}^{k} \frac{1}{\Upsilon^{(j)}} & , \quad k < i \\[2ex] \dfrac{-\left[\chi_{\xi\gamma}^{(i)}\varphi_\xi^{(i)} - \chi_{\xi\xi}^{(i)}\varphi_\gamma^{(i)}\right] t_{go_i}}{\eta_i^2 \chi_{\xi\xi}^{(i)} \Upsilon^{(i)} \Delta_\infty V_i} \displaystyle\sum_{j=k+1}^{n} \frac{1}{\Upsilon^{(j)}} & , \quad k \geq i \end{cases}$$

Optimal Cooperative Law - Properties

Assuming identical closed-loop dynamics for all team members we obtain

$$\varphi_\xi^{(i)}(t_{go}) = k_i \varphi_\xi(t_{go}) \quad , \quad \varphi_\xi(t_{go}) = \mathbf{D}_\xi \bar{\mathbf{\Phi}}(t_{go}) \bar{\mathbf{B}} \tag{35a}$$

$$\varphi_\gamma^{(i)}(t_{go}) = \varphi_\gamma(t_{go})/V_i \quad , \quad \varphi_\gamma(t_{go}) = \mathbf{D}_\gamma \bar{\mathbf{\Phi}}(t_{go}) \bar{\mathbf{B}} \tag{35b}$$

where $\bar{\mathbf{\Phi}}(t_{go})$ and $\bar{\mathbf{B}}$ are the transition and control matrices, and

$$\chi_{\xi\xi}^{(i)}(t_{go}) = \frac{k_i^2}{\eta_i^2}\chi_{\xi\xi}(t_{go}),\ \chi_{\xi\gamma}^{(i)}(t_{go}) = \frac{k_i}{\eta_i^2 V_i}\chi_{\xi\gamma}(t_{go}),\ \chi_{\gamma\gamma}^{(i)}(t_{go}) = \frac{1}{\eta_i^2 V_i^2}\chi_{\gamma\gamma}(t_{go}) \tag{36}$$

$\chi_{\xi\xi}(t_{go}), \chi_{\xi\gamma}(t_{go})$, and $\chi_{\gamma\gamma}(t_{go})$ are

$$\chi_{jl}(t_{go}) = \int_0^{t_{go}} \varphi_j(\tau)\varphi_l(\tau)d\tau \quad , \quad j,l \in \{\xi,\gamma\} \tag{37}$$

Optimal Cooperative Law - Properties

This yields

$$N_{Z_k}^{u_i} = \begin{cases} \dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)\chi_{\xi\gamma}t_{go}^2}{k_i\chi_{\xi\xi}(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi})} \dfrac{\left(\sum_{j=1}^n V_j^2\eta_j^2 - V_i^2\eta_i^2\right)}{\sum_{j=1}^n V_j^2\eta_j^2} - \dfrac{\varphi_\xi t_{go}^2}{k_i\chi_{\xi\xi}} & , k = i \\ -\dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)\chi_{\xi\gamma}t_{go}^2}{k_k\chi_{\xi\xi}\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)} \dfrac{V_k V_i \eta_k^2}{\sum_{j=1}^n V_j^2\eta_j^2} & , k \neq i \end{cases} \tag{38}$$

$$N_{\Delta Z_{n+k}}^{u_i} = \begin{cases} \dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)t_{go}}{\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)} \dfrac{\sum_{j=1}^k V_j^2\eta_j^2}{\sum_{j=1}^n V_j^2\eta_j^2} & , \ k < i \\ -\dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)t_{go}}{\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)} \dfrac{\sum_{j=k+1}^n V_j^2\eta_j^2}{\sum_{j=1}^n V_j^2\eta_j^2} & , \ k \geq i \end{cases} \tag{39}$$

Optimal Cooperative Law - Properties

Assuming equal team members speeds $V_i = V$ and maneuver capabilities $\eta_i = 1,\ \forall i$ yields navigation gains as a function of the team's size

$$N_{Z_k}^{u_i} = \begin{cases} \dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)\chi_{\xi\gamma}t_{go}^2}{k_i\chi_{\xi\xi}\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)}\dfrac{(n-1)}{n} - \dfrac{\varphi_\xi t_{go}^2}{k_i\chi_{\xi\xi}} & , \ k = i \\ \dfrac{-\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)\chi_{\xi\gamma}t_{go}^2}{k_k\chi_{\xi\xi}\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)}\dfrac{1}{n} & , \ k \neq i \end{cases} \tag{40}$$

$$N_{\Delta Z_{n+k}}^{u_i} = \begin{cases} \dfrac{\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)t_{go}}{\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)}\dfrac{k}{n} & , \ k < i \\ \dfrac{-\left(\chi_{\xi\gamma}\varphi_\xi - \chi_{\xi\xi}\varphi_\gamma\right)t_{go}}{\left(\chi_{\xi\gamma}^2 - \chi_{\gamma\gamma}\chi_{\xi\xi}\right)}\dfrac{(n-k)}{n} & , \ k \geq i \end{cases} \tag{41}$$

Optimal Cooperative Law - Properties

- The ZEM navigation gains are equally dependent on all the other missiles in the team.
- In contrast, the ZEA navigation gains are linearly dependent on the index distance $|i - k|$.
- Choosing $n \to \infty$ results in convergence of the ZEM navigation gains to the 1-on-1 ZEM navigation gains and the ZEA navigation gains are bounded by the 1-on-1 ZEA navigation gains.

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2-on-1 Solution

Simplifying assumptions for analytic solution: 2-on-1, $a_T = const$, ideal dynamics

Cost function

$$J = \frac{\alpha_1}{2} Z_1^2(t_{f2}) + \frac{\alpha_2}{2} Z_2^2(t_{f2}) + \frac{\beta}{2}[Z_3(t_{f2}) - Z_4(t_{f2}) - \Delta_{c1}]^2 + \frac{1}{2}\int_0^{t_{f2}} \eta_1^2 u_1^2 + \eta_2^2 u_2^2 dt \quad (42)$$

Zero effort miss dynamics

$$\begin{cases} \dot{Z}_1 = -k_1(t_{f1} - t)u_1 \\ \dot{Z}_2 = -k_2(t_{f2} - t)u_2 \\ \dot{Z}_3 = u_1/V_1 \\ \dot{Z}_4 = u_2/V_2 \end{cases} \quad (43)$$

2-on-1 Solution

$$u_1^*(t) = \frac{N'^{u_1}_{Z_1}}{t_{go1}^2} Z_1(t) + \frac{N'^{u_1}_{Z_2}}{t_{go2}^2} Z_2(t) + N'^{u_1}_{Z_{34}} \frac{V_1}{t_{go1}} (Z_3(t) - Z_4(t) - \Delta_c) \tag{44a}$$

$$u_2^*(t) = \frac{N'^{u_2}_{Z_1}}{t_{go1}^2} Z_1(t) + \frac{N'^{u_2}_{Z_2}}{t_{go2}^2} Z_2(t) + N'^{u_2}_{Z_{34}} \frac{V_2}{t_{go2}} (Z_3(t) - Z_4(t) - \Delta_c) \tag{44b}$$

where

$$Z_1(t) = \xi_1 + \dot{\xi}_1 t_{go1} + k_{T1} a_T t_{go1}^2/2 \tag{45a}$$

$$Z_2(t) = \xi_2 + \dot{\xi}_2 t_{go2} + k_{T2} a_T t_{go2}^2/2 \tag{45b}$$

$$Z_3(t) = t_{go1} a_T/V_T + \gamma_T + \gamma_1 \tag{45c}$$

$$Z_4(t) = t_{go2} a_T/V_T + \gamma_T + \gamma_2 \tag{45d}$$

2-on-1 Solution

The navigation gains are

$$N'^{u_1}_{Z_1} = 3k_1 t_{go1}^3 \alpha_1 \left[t_{go2} V_1^2 \beta C_{22} + 2V_2^2 \eta^2 C_{21} \left(2V_1^2 + \beta t_{go1}\right)\right] / \Delta_z \tag{46a}$$

$$N'^{u_1}_{Z_2} = 3k_2 t_{go2}^4 V_1 V_2 \beta \alpha_2 \eta^2 \left(6 - k_1^2 t_{go1}^3 \alpha_1\right) / \Delta_z \tag{46b}$$

$$N'^{u_1}_{Z_{34}} = 2V_2^2 \beta \eta^2 t_{go1} C_{21} \left(k_1^2 t_{go1}^3 \alpha_1 - 6\right) / \Delta_z \tag{46c}$$

$$N'^{u_2}_{Z_1} = 3k_1 t_{go1}^4 V_1 V_2 \beta \alpha_1 \left(6\eta^2 - k_2^2 t_{go2}^3 \alpha_2\right) / \Delta_z \tag{46d}$$

$$N'^{u_2}_{Z_2} = 3k_2 t_{go2}^3 \alpha_2 \left[t_{go1} V_2^2 \beta \eta^2 C_{12} + 2V_1^2 C_{11} \left(2V_2^2 \eta^2 + \beta t_{go2}\right)\right] / \Delta_z \tag{46e}$$

$$N'^{u_2}_{Z_{34}} = 2V_1^2 \beta t_{go2} C_{11} \left(6\eta^2 - k_2^2 t_{go2}^3 \alpha_2\right) / \Delta_z \tag{46f}$$

2-on-1 Solution

By enforcing perfect intercept and intercept angle ($\alpha_1, \alpha_2, \beta \to \infty$), and assuming an identical missile team launched at the same time ($V_1 = V_2 = V$, and $\eta = 1$) we obtain

$$N_{Z_1}^{u_1} = \frac{9}{2k_1} \quad , \quad N_{Z_2}^{u_1} = \frac{-3}{2k_2} \quad , \quad N_{\Delta Z_3}^{u_1} = 1 \tag{47}$$

By dictating that the second missile does not maneuver ($\eta^2 \to \infty$), the 1-on-1 optimal guidance law is obtained with

$$N_{Z_1}^{u_1} = \frac{6}{k_1} \quad , \quad N_{\Delta Z_3}^{u_1} = 2 \tag{48}$$

The absolute values of $N_{Z_1}^{u_1}$ and $N_{\Delta Z_3}^{u_1}$ for the cooperative case are bounded from above by the 1-on-1 gains, and $N_{Z_1}^{u_1}$ is slightly higher than the optimal APN gain ($N_{Z_1}^{u_i} = 3/k_1$). These properties actually hold for any team size (ideal dynamics case).

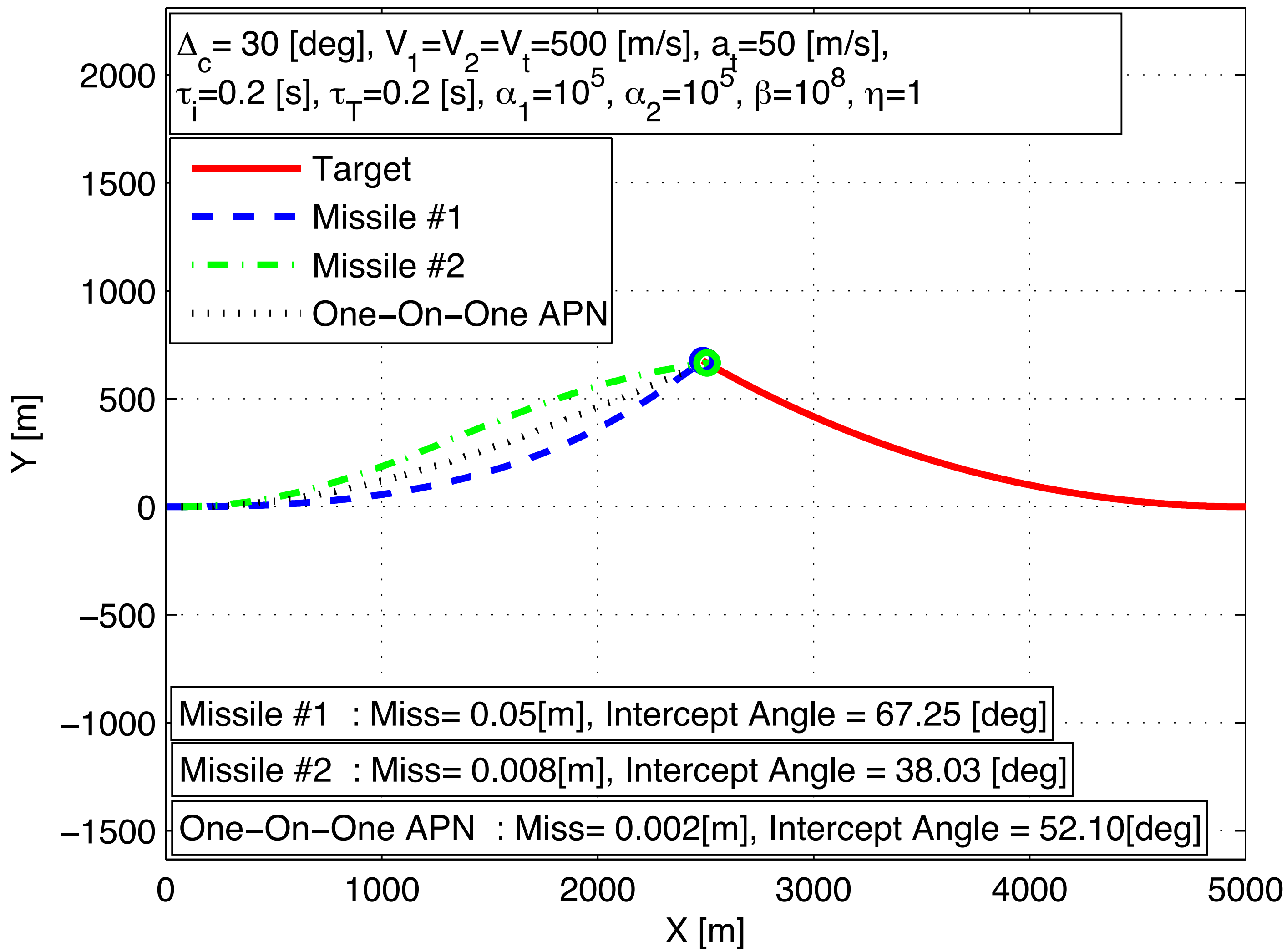
Outline

- $\surd$ Introduction
- $\surd$ Mathematical Model
- $\surd$ Optimization Problem
- $\surd$ Optimal Cooperative Law
- $\surd$ 2-on1 Solution
- **Performance Analysis**
- Summary & Conclusions

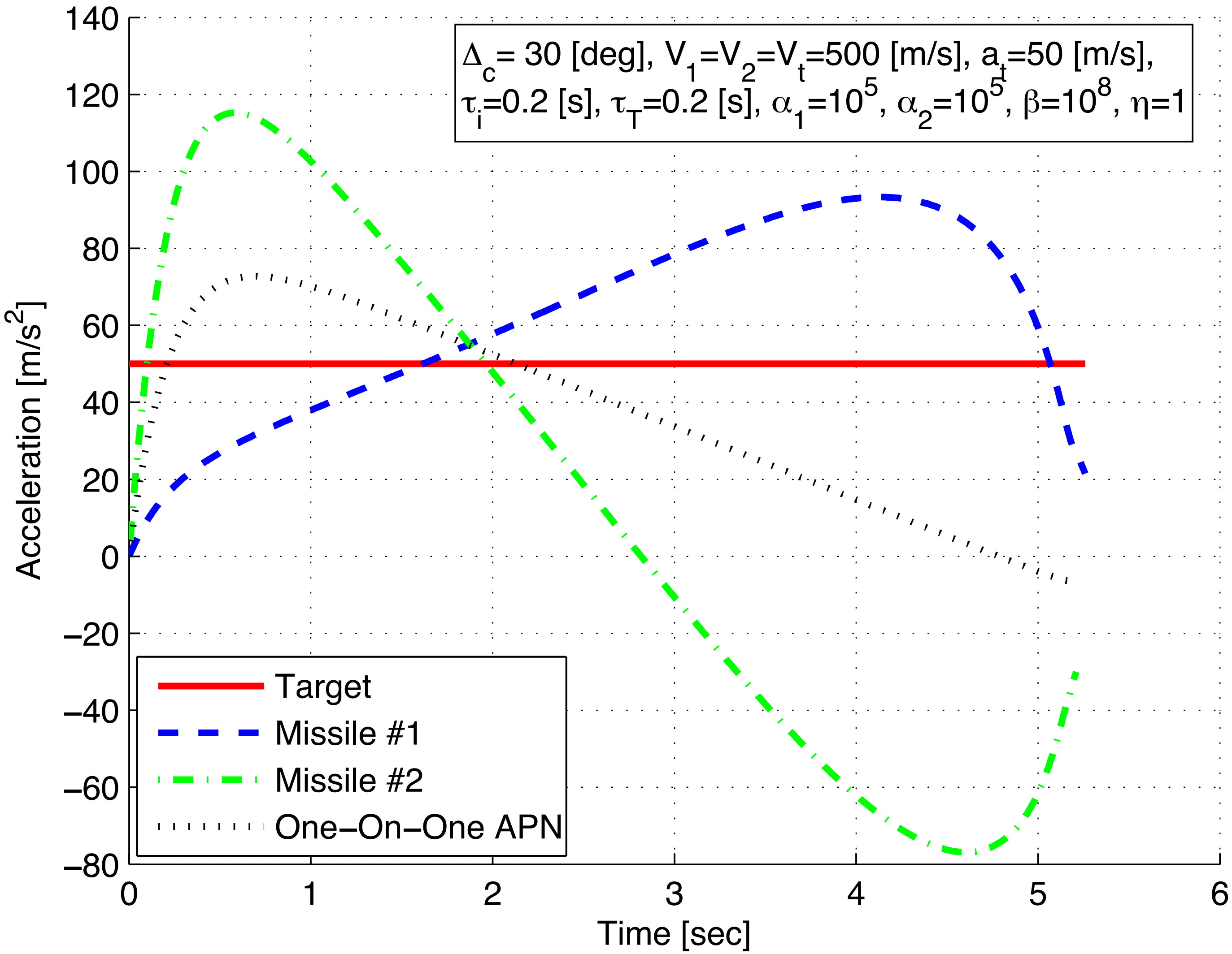
Performance Analysis

Parameter	Value	units
Δ	30	deg
ρ_0	5000	$[m]$
V_T	500	$[m/s]$
V_1, V_2	500	$[m/s]$
τ_T	0.2	$[sec]$
τ_1, τ_2	0.2	$[sec]$
a_T	50	$[m/s^2]$

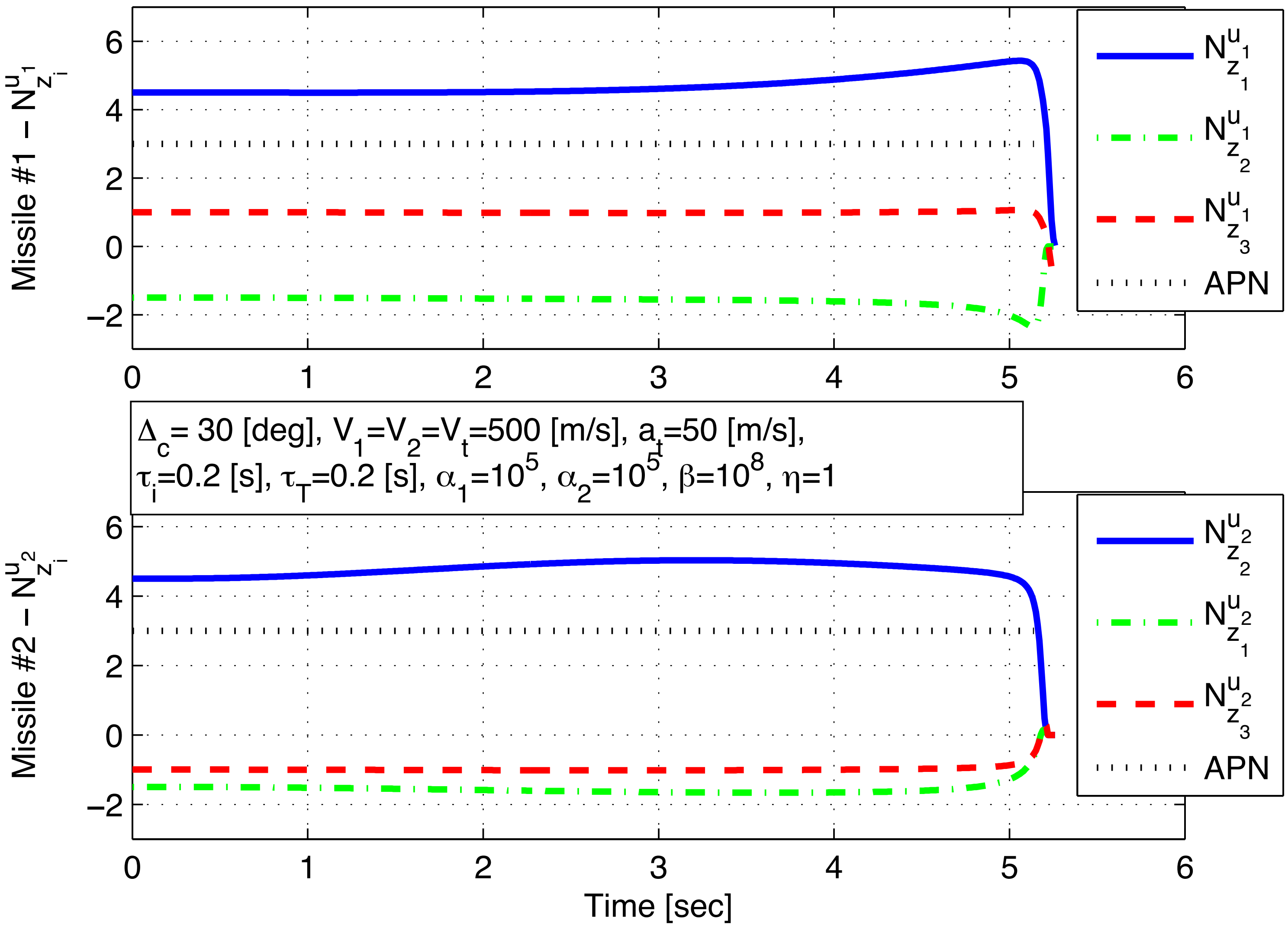
Explicit Cooperation - Trajectories



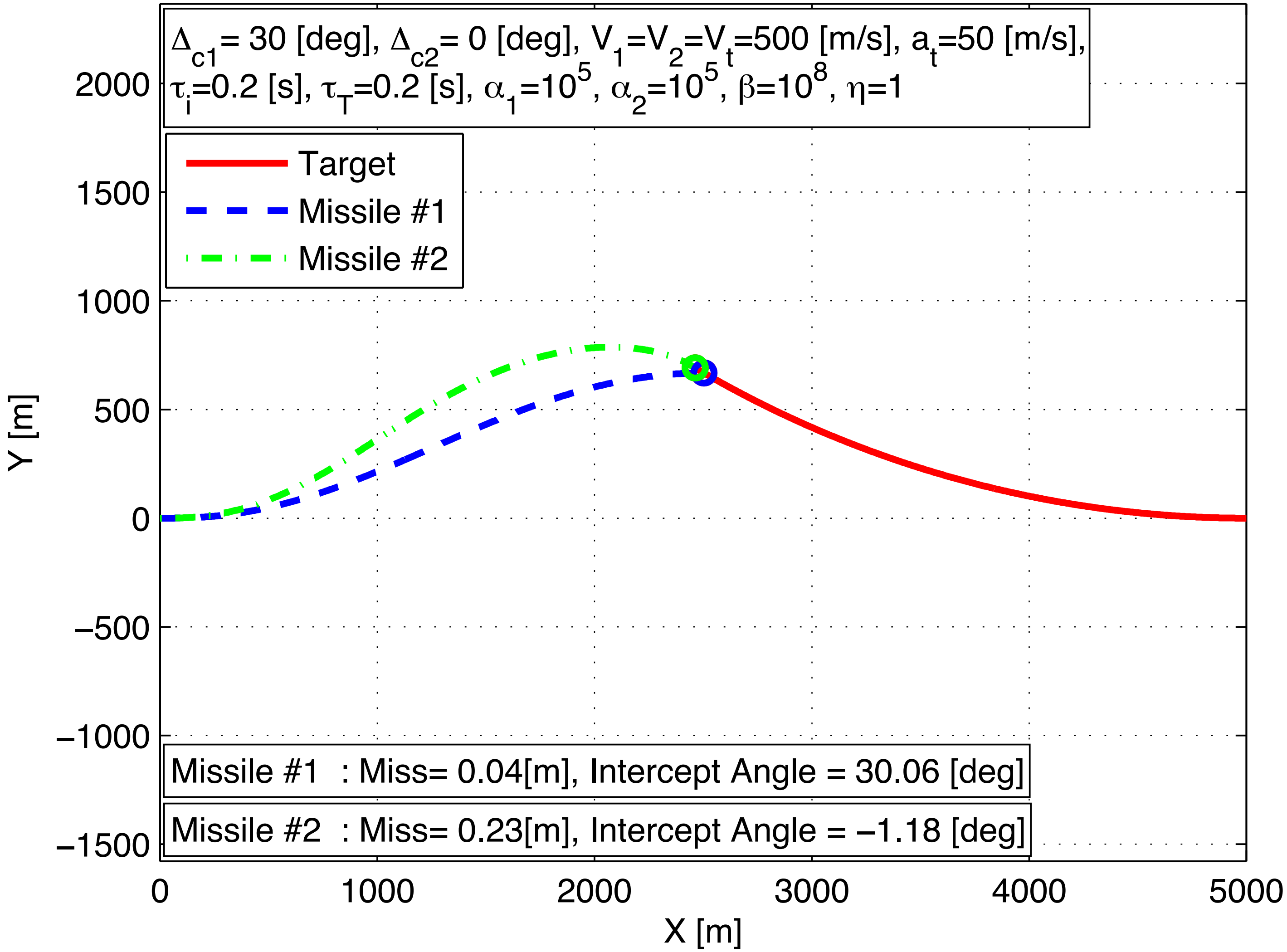
Explicit Cooperation - Accelerations



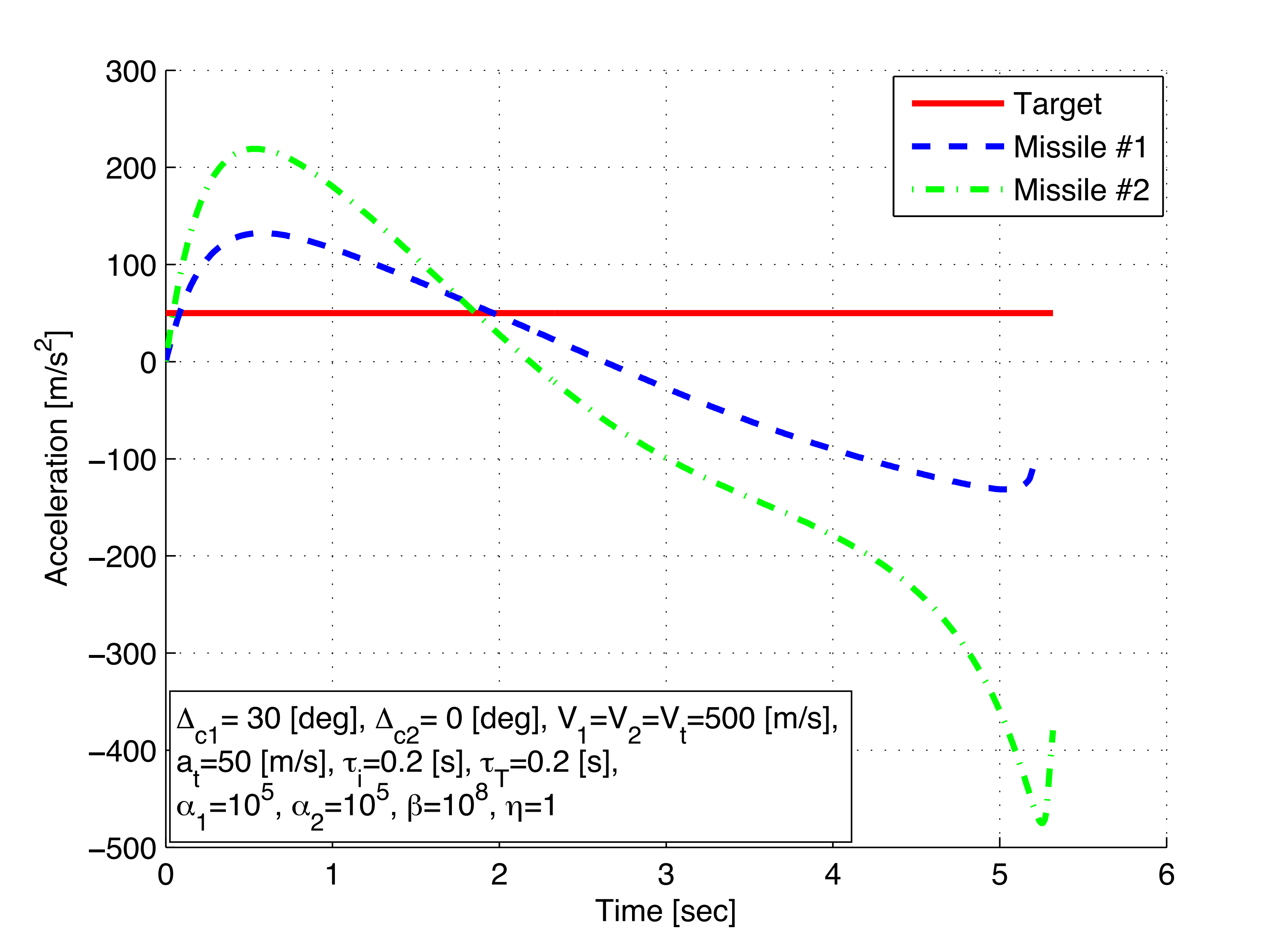
Explicit Cooperation - Navigation Gains



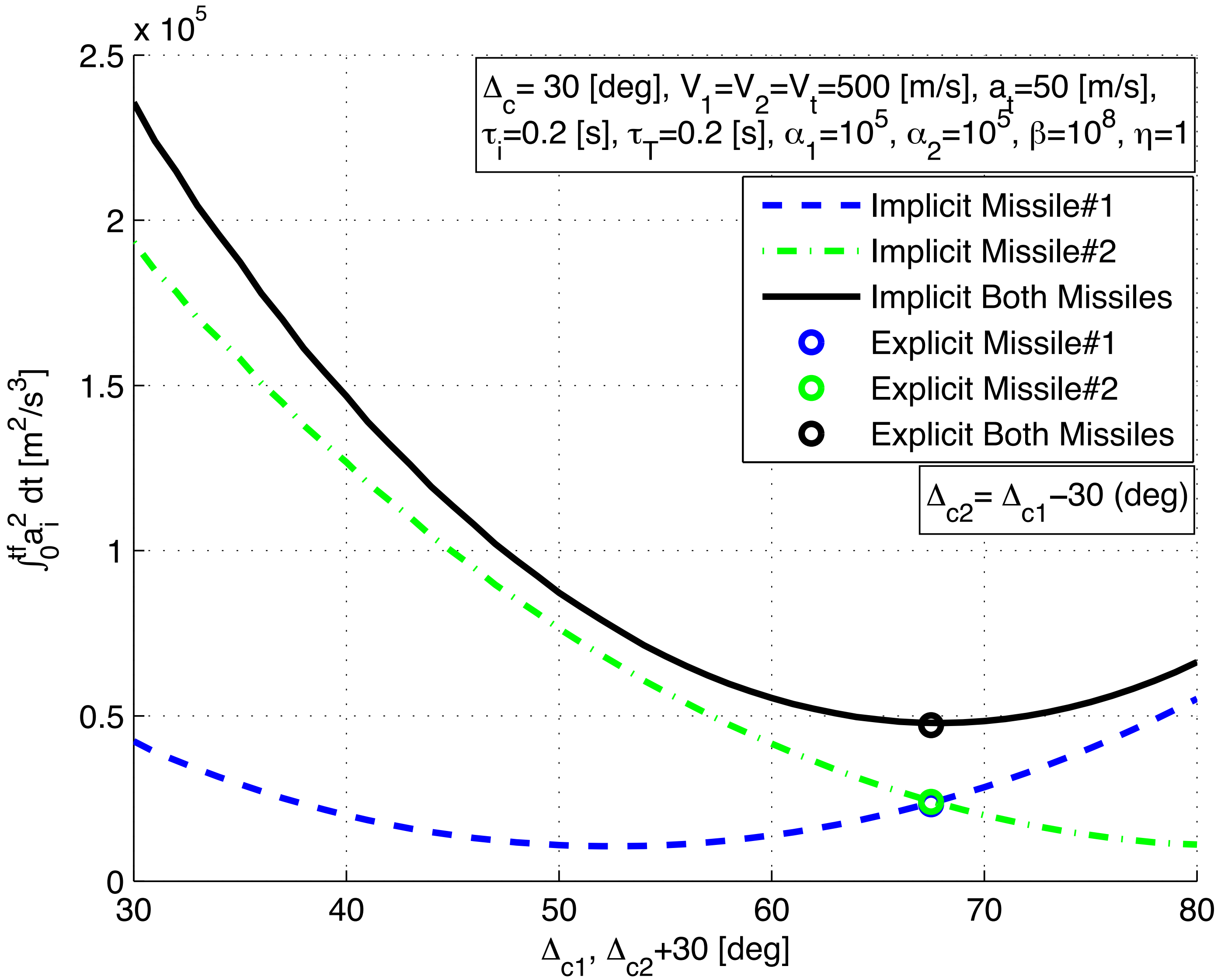
Implicit Cooperation - Trajectories (0,30)



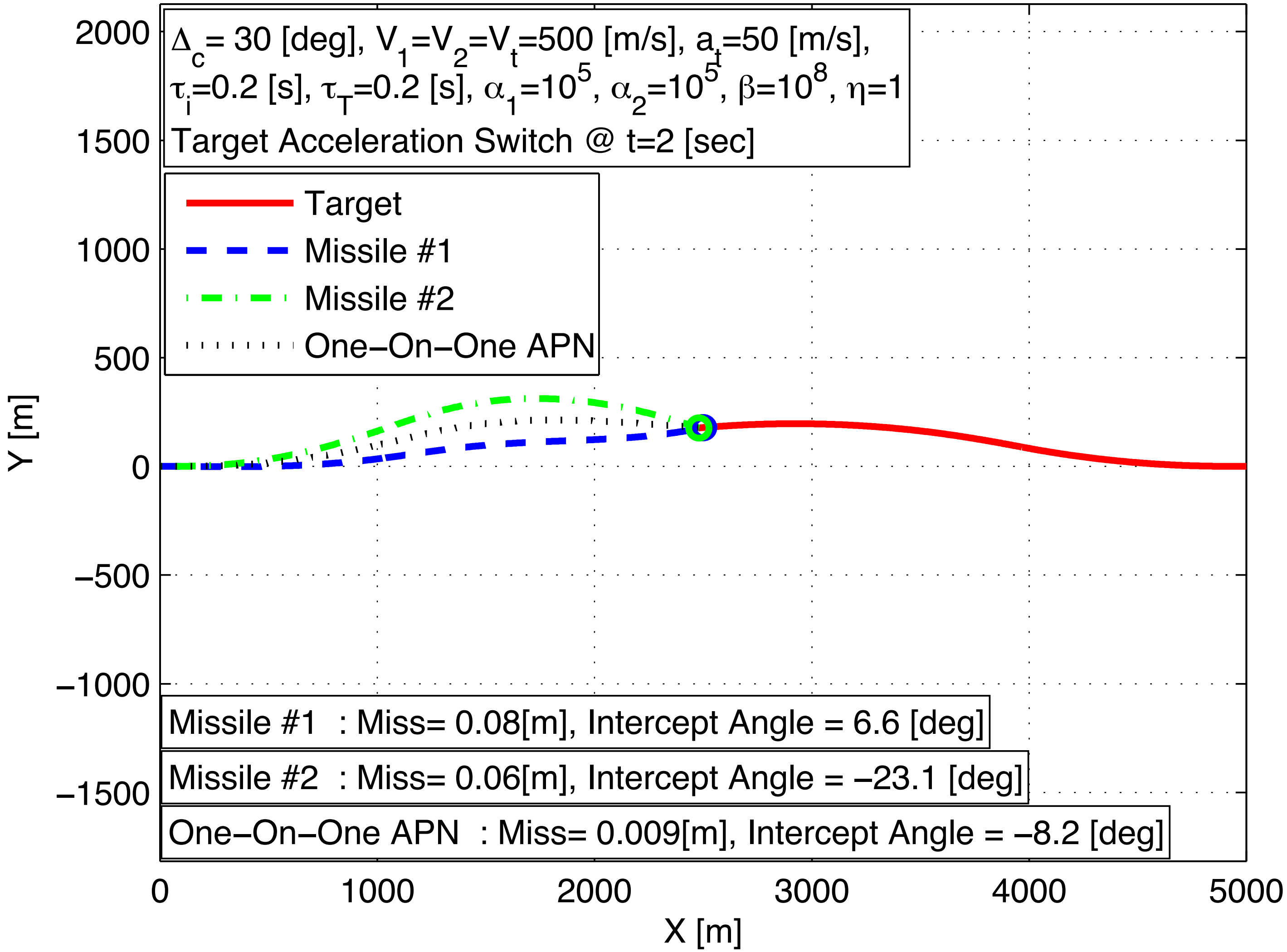
Implicit Cooperation - Accelerations (0,30)



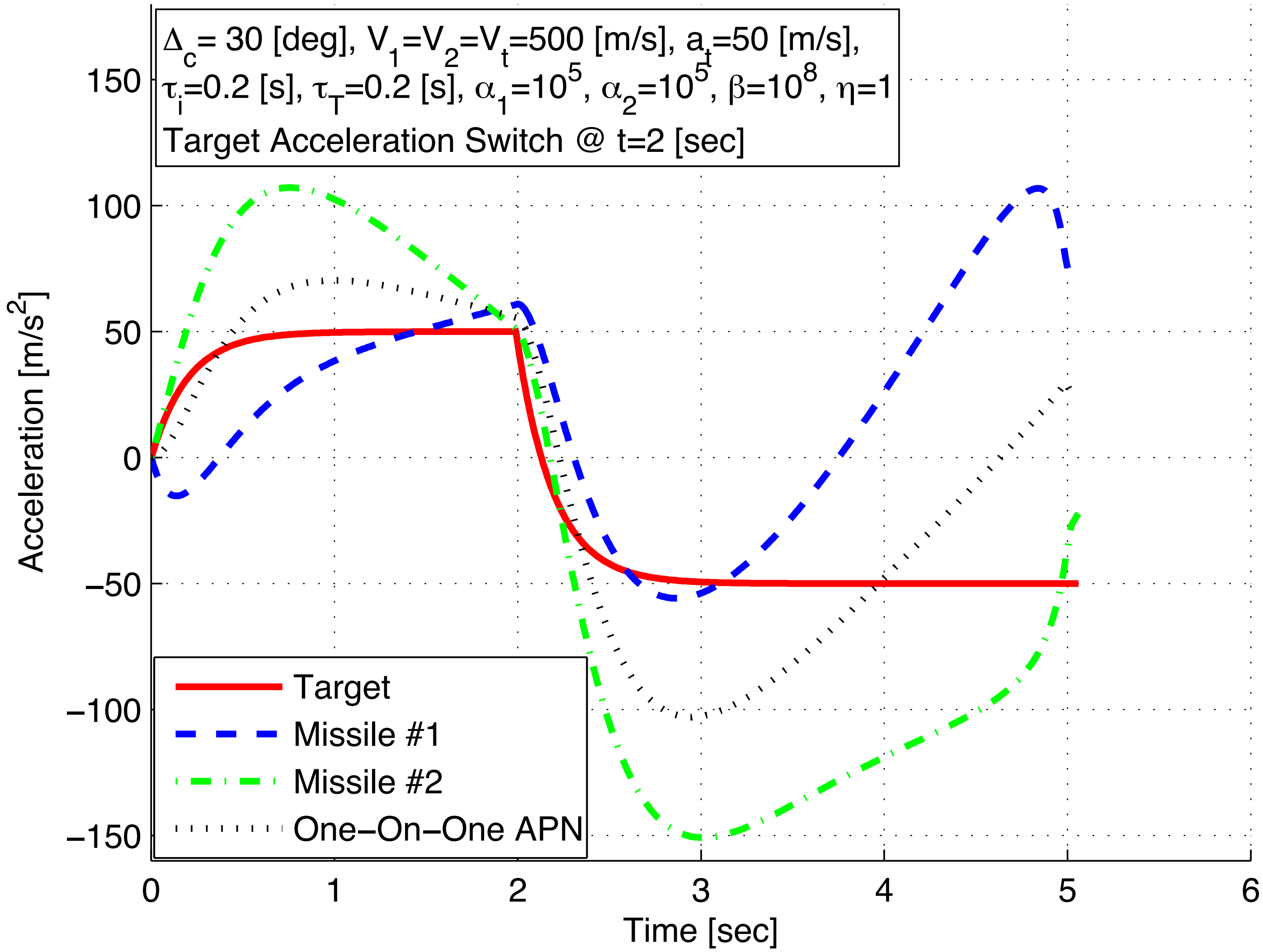
Comparison Between Explicit & Implicit Cooperation



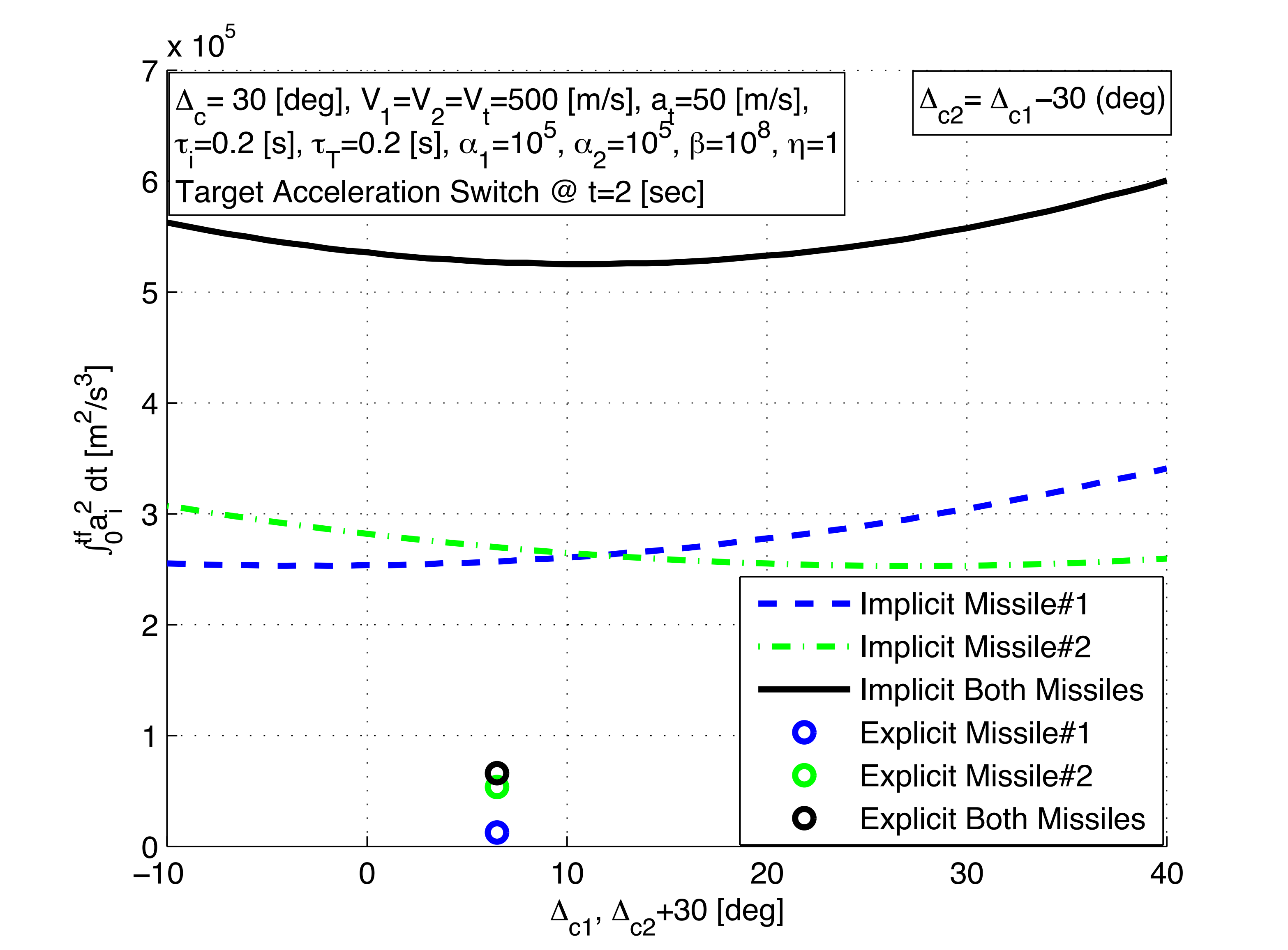
Robustness to Target Maneuvers - Trajectories



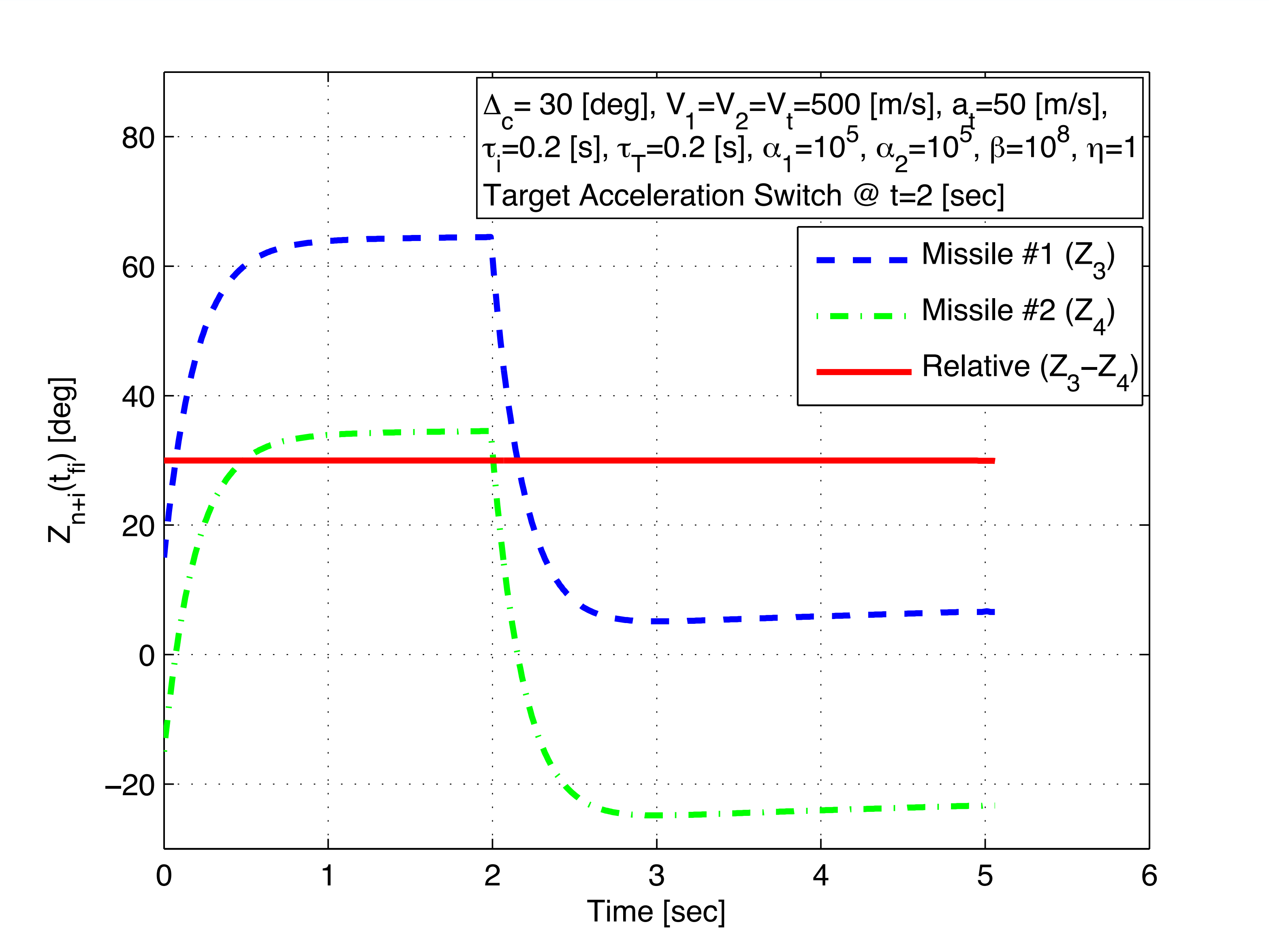
Robustness to Target Maneuvers - Accelerations



Robustness to Target Maneuvers - Comparison



Robustness to Target Maneuvers - Adaptation



Summary & Conclusions

- Explicit cooperative guidance laws have been presented
- Compared to, 1-on-1 based, implicit guidance laws
- Explicit cooperation is much superior to implicit cooperation
- Cooperation dramatically improved homing performance and reduced control effort
- Relative intercept angle capability can be used for saturating target defences, improve observability, etc.
- Research was partially supported by the Israel Science Foundation
- Shaferman, V. and Shima, T., “Cooperative Optimal Guidance Laws for Imposing a Relative Intercept Angle" *AIAA Journal Guidance, Control, and Dynamics*, 2014.