GSC'17 - Graduate Students in Control

Formulation of Optimal Semi-Active Feedback by Krotov's Method

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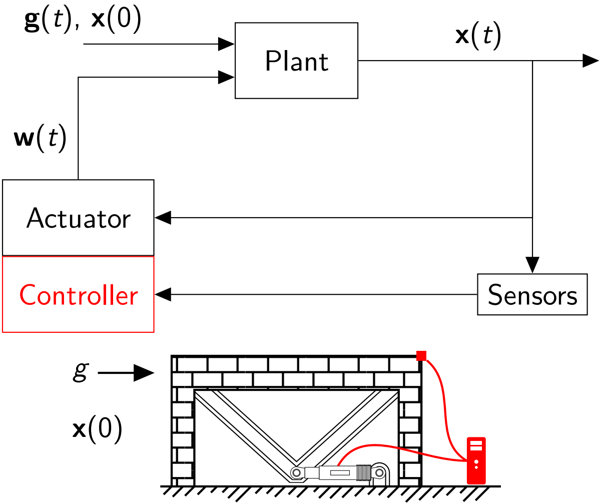
- Vibration control.
- Structural control - a branch of vibration control. Exploits control theory to enhance dynamic response of structures. Mostly those induced by winds, earthquakes and man.

Ji-Ji earthquake
Taiwan, 1999

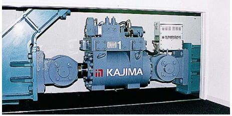
Man induced vibration

- Common structural control realization consists of mechanical actuators which apply forces to the vibrating structure.

A simplified block diagram of a controlled structure:

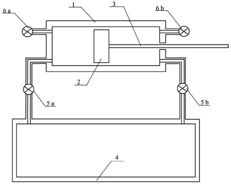


- Semi-active dampers are type of actuators which are characterized by low energy consumption, dissipativity and inherent stability.

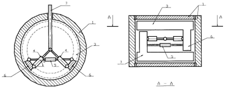


Variable Orifice

MR



Pneumatic



Variable Viscous

A possible approach for semi-active control design:

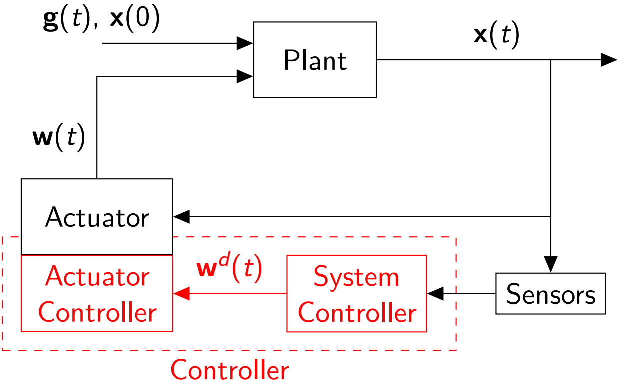
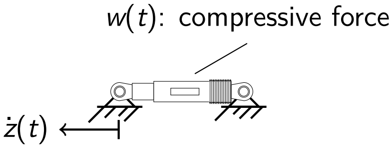


Figure: A semi active controlled system

If $\mathbf{w}(t) = \mathbf{w}^d(t) \forall t$, than $\mathbf{w}^d$ is *realizable*.
The significance of a realizable $\mathbf{w}^d$ increases when optimal control is desired.

In order to compute a realizable $\mathbf{w}^d$, the system controller must consider the actuator constraints.
For semi-active controlled plants, it must consider a fundamental semi-active constraint.



Definition 1 (Semi-Active Constraint).

$$w(t)\dot{z}(t) \leq 0$$

or

$$w(t)\mathbf{cx}(t) \leq 0$$

To compute a realizable optimal feedback for mechanical plant, which is controlled by a single semi-active damper.
The solution consists of two key stages:

1. Reformulation of the problem by writing the linear state equation as an equivalent bilinear one.
2. Using Krotov's method to derive an algorithm for the computation of an optimal semi-active feedback.

The DOF model for free vibrating linear structure, equipped with a single actuator:

$$\mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{C}_d\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \phi w(t); z(0), \dot{z}(0) \in \mathbb{R}^{n_z}, \forall t \in (0, t_f) \tag{1}$$

where $\mathbf{M} > 0$, $\mathbf{C}_d \geq 0$ and $\mathbf{K} > 0$ are $n_z \times n_z$; n_z is the number of dynamic degrees of freedom (DOF); $\mathbf{z} : [0, t_f] \to \mathbb{R}^{n_z}$ is a smooth vector function of the DOF displacements; $w : [0, t_f] \to \mathbb{R}$ is a control force signal and $\phi \in \mathbb{R}^{n_z}$ is an input vector that describes how the control force affects the structure's DOF. The state space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}w(t); \quad \mathbf{x}(0); \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} \tag{2a}$$

$$\mathbf{A} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}_d \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{2b}$$

$$\mathbf{b} \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\phi \end{bmatrix} \in \mathbb{R}^{n} \tag{2c}$$

By writing **w** in the bilinear form:

$$w(t) = \hat{w}(t, \mathbf{x}(t)) = -u(t)\mathbf{c}\mathbf{x}(t); \quad u(t) \geq 0$$

we are assured that the semi-active constraint is intrinsically satisfied.

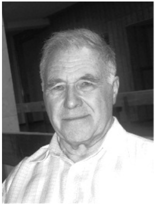
Definition 2 (CBQR).

A *constrained bilinear quadratic regulator* refers to the minimization of

$$J(\mathbf{x}, u) = \frac{1}{2} \int_0^{t_f} \mathbf{x}(t)^T \mathbf{Q} \mathbf{x}(t) + r u(t)^2 \mathrm{d}\, t; \quad \mathbf{Q} \geq 0, r > 0$$

subjected to

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - u(t)\mathbf{b}\mathbf{c}]\mathbf{x}(t), \ \mathbf{x}(0); \qquad u(t) \geq 0$$



V. F. Krotov (1932-2015)

- Starting in the sixties, sufficient conditions for global optimum of optimal control problems were published by V. F. Krotov [1] .
- It enabled the computation of a global optimum by an algorithm which is known as *global method of successive improvements of control* or *Krotov's method*.

[1] V. F. Krotov, *Global Methods in Optimal Control Theory*, 1995.

Let

- $\mathscr{U}$: A set of admissible control signals.
- $\mathscr{X}$: A linear space of state vector functions.
- admissible process: A pair $(\mathbf{x}, \mathbf{u})$, where $\mathbf{u} \in \mathscr{U}$, $\mathbf{x} \in \mathscr{X}$ and they both satisfy the states equation

$$\begin{aligned} \dot{\mathbf{x}}(t) =& \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ & \mathbf{x}(0) \in \mathbb{R}^n,\ t \in (0, t_f) \end{aligned} \tag{3}$$

- $\mathscr{X}(t)$: The set $\{\mathbf{x}(t) | \mathbf{x} \in \mathscr{X}\} \subset \mathbb{R}^n$, i.e., an intersection of $\mathscr{X}$ at a given t. For instance, $\mathscr{X}(t_f) \subseteq \mathbb{R}^n$ is a set of all the terminal states of the processes in $\mathscr{X}$.

- *q*: A piecewise smooth function $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, denoted as *Krotov function* or *solving function*. Its partial derivatives are denoted by q_t and $q_{\mathbf{x}}$.
- *J*: A performance index. It is a functional

$$J(\mathbf{x}, \mathbf{u}) = l_f(\mathbf{x}(t_f)) + \int_0^{t_f} l(\mathbf{x}(t), \mathbf{u}(t), t) \mathrm{d}\, t$$

  where $l_f : \mathbb{R}^n \to \mathbb{R}$ and $l : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$ are continuous.

Each q is related with an equivalent formulation of performance index, as follows.

Theorem 3.

Let $(\mathbf{x}, \mathbf{u})$ *be an admissible process. For each* q *there is an equivalent representation of* $J(\mathbf{x}, \mathbf{u})$*:*

$$J_{eq}(\mathbf{x}, \mathbf{u}) = s_f(\mathbf{x}(t_f)) + q(\mathbf{x}(0), 0) + \int_0^{t_f} s(\mathbf{x}(t), \mathbf{u}(t), t) \mathrm{d}\, t \equiv J(\mathbf{x}, \mathbf{u}) \tag{4}$$

$$s(\mathbf{x}(t), \mathbf{u}(t), t) \triangleq q_t(\mathbf{x}(t), t) + q_{\mathbf{x}}(\mathbf{x}(t), t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + l(\mathbf{x}(t), \mathbf{u}(t), t) \tag{5a}$$

$$s_f(\mathbf{x}(t_f)) \triangleq l_f(\mathbf{x}(t_f)) - q(\mathbf{x}(t_f), t_f) \tag{5b}$$

Proof.

Substitute s and s_f in J_{eq}, and then use Newton-Leibniz formula. □

- It should be stressed that $J(\mathbf{x}, \mathbf{u}) = J_{eq}(\mathbf{x}, \mathbf{u})$ holds if **x** and **u** satisfy the state equation (Eq. (3)).
- As q is not unique, the equivalent representation J_{eq}, s and s_f, are also non-unique.
- In many publications, s is written with an opposite sign before l, Though, there is no intrinsic difference between these two formulations.

Sufficient condition for a global optimal admissible process, by means of J_{eq}, is given by:

Theorem 4.

Let s and s_f be related with some q. Let $(\mathbf{x}^*, \mathbf{u}^*)$ be an admissible process. If:

$$\begin{aligned} s(\mathbf{x}^*(t), \mathbf{u}^*(t), t) &= \min_{\substack{\mathbf{x}(t)\in\mathscr{X}(t) \\ \mathbf{u}(t)\in\mathscr{U}(t)}} s(\mathbf{x}(t), \mathbf{u}(t), t) \quad \forall t \in [0, t_f) \\ s_f(\mathbf{x}^*(t_f)) &= \min_{\mathbf{x}(t_f)\in\mathscr{X}(t_f)} s_f(\mathbf{x}(t_f)) \end{aligned} \tag{6}$$

then $(\mathbf{x}^*, \mathbf{u}^*)$ is an optimal process.

Proof.

Assume that Eq. (6) holds and expand the difference $J_{eq}(\mathbf{x}, \mathbf{u}) - J_{eq}(\mathbf{x}^*, \mathbf{u}^*)$. □

- An optimum derived by theorem 4 is global since the minimization problem defined in Eq. (6) is global.
- Theorems 4 and 3 provide a hint for finding a global optimum. That is, one should formulate q such that it will be possible to compute $(\mathbf{x}^*, \mathbf{u}^*)$ from:

$$s(\mathbf{x}^*(t), \mathbf{u}^*(t), t) = \min_{\substack{\mathbf{x}(t)\in\mathscr{X}(t)\\ \mathbf{u}(t)\in\mathscr{U}(t)}} s(\mathbf{x}(t), \mathbf{u}(t), t) \quad \forall t \in [0, t_f)$$

$$s_f(\mathbf{x}^*(t_f)) = \min_{\mathbf{x}(t_f)\in\mathscr{X}(t_f)} s_f(\mathbf{x}(t_f))$$

However, the main problem remains - the existence and formulation of such q. Note that a similar approach is used in Lyapunov's method of stability.

- Krotov's sufficient condition lays the foundation for novel algorithms for the solution of optimal control problems. Krotov's method is one of them.
- According to this method, the solution is not direct but a sequential one. It yields a sequence of admissible processes which converges monotonically to an optimum $(\mathbf{x}^*, \mathbf{u}^*)$. Such a sequence of processes is called an *optimizing sequence*:

$$(\mathbf{x}_k, \mathbf{u}_k) \to (\mathbf{x}^*, \mathbf{u}^*); \quad J(\mathbf{x}_k, \mathbf{u}_k) \geq J(\mathbf{x}_{k+1}, \mathbf{u}_{k+1})$$

s.t.

$$\dot{\mathbf{x}}^*(t) = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t); \quad \mathbf{u}^* = \arg\min_{\mathbf{x},\mathbf{u}} J(\mathbf{x}, \mathbf{u})$$

The use of Krotov's method relies on the solution of another **key problem** which is the formulation of a sequence of solving functions - $\{q_k\}$.

- $\mathbf{u}$: A control signal, i.e. a mapping $\mathbb{R} \to \mathbb{R}^{n_u}$.
- $\hat{\mathbf{u}}$: A control law, i.e. a mapping $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n_u}$.

Let $(\mathbf{x}_0, \mathbf{u}_0)$ be some initial admissible process. An improved process $(\mathbf{x}_1, \mathbf{u}_1)$ is computed in the following manner:

1. Formulate q_0 such that s_0 and s_{f0} will satisfy:

$$s_0(\mathbf{x}_0(t), \mathbf{u}_0(t), t) = \max_{\mathbf{x} \in \mathscr{X}(t)} s_0(\mathbf{x}, \mathbf{u}_0(t), t) \ \forall t \in [0, t_f)$$

$$s_{f0}(\mathbf{x}_0(t_f)) = \max_{\mathbf{x} \in \mathscr{X}(t_f)} s_{f0}(\mathbf{x})$$

2. Formulate a control law $\hat{\mathbf{u}}_0$ such that

$$\hat{\mathbf{u}}_0(\mathbf{x}(t), t) = \arg \min_{\mathbf{u} \in \mathscr{U}(t)} s_0(\mathbf{x}(t), \mathbf{u}, t) \ \forall \mathbf{x} \in \mathscr{X}, t \in [0, t_f]$$

3. Solve $\dot{\mathbf{x}}_1(t) = \mathbf{f}(\mathbf{x}_1(t), \hat{\mathbf{u}}_0(\mathbf{x}_1(t), t), t)$, for the given $\mathbf{x}(0)$, and set $\mathbf{u}_1(t) = \hat{\mathbf{u}}_0(\mathbf{x}_1(t), t)$ for all $t \in [0, t_f]$.

$(\mathbf{x}_2, \mathbf{u}_2)$ is computed by starting over from $(\mathbf{x}_1, \mathbf{u}_1)$, formulation of q_1 and $\hat{\mathbf{u}}_1$, and so on.

- The use of Krotov's method is not straightforward. It requires the formulation of a suitable sequence $\{q_k\}$, such that s_k and s_{fk} will satisfy the aforementioned min/max problem.
- The search for such a sequence is a significant challenge. There is no known unified approach for formulating Krotov functions and they usually differ from one optimal control problem to another.
- In this work, a suitable sequence of Krotov functions was found for the defined CBQR problem.

The next theorem defines the CBQR control law (step 2 in Krotov's method).

Theorem 5.

Let the Krotov function be

$$q(\mathbf{x}(t), t) = 0.5\mathbf{x}(t)^T\mathbf{P}(t)\mathbf{x}(t); \quad \mathbf{P}(t_f) = \mathbf{0} \tag{7}$$

where $\mathbf{P} : \mathbb{R} \to \mathbb{R}^{n\times n}$ *is a symmetric matrix function, smooth above* $(0, t_f)$*; and let* $\mathbf{x}$ *be a given process. There exists a unique control law that minimizes* $s(\mathbf{x}(t), u(t), t)$ *and* $s_f(\mathbf{x}(t_f))$*. It is defined by:*

$$\hat{u}(\mathbf{x}(t), t) = \max\left\{\frac{\mathbf{x}(t)^T\mathbf{P}(t)\mathbf{bcx}(t)}{r}, 0\right\} \tag{8}$$

Proof.

By substitution of Eq. (7) in Eq. (5a) and (5b); completing the square and minimizing with relation to an admissible $u(t)$. □

The next theorem defines q_k that corresponds $(\mathbf{x}_k, \mathbf{u}_k)$ (step 1 in Krotov's method).

Theorem 6.

Let $(\mathbf{x}_k, u_k)$ *be a given admissible process and let* $\mathbf{P}_k$ *be the solution of:*

$$\begin{aligned} \dot{\mathbf{P}}_k(t) &= -\,\mathbf{P}_k(t)[\mathbf{A} - u_k(t)\mathbf{bc}] - [\mathbf{A}^T - u_k(t)\mathbf{c}^T\mathbf{b}^T]\mathbf{P}_k(t) - \mathbf{Q} \\ \mathbf{P}_k(t_f) &= \mathbf{0};\;\; t \in (0, t_f) \end{aligned} \tag{9}$$

The Krotov function $q_k(\mathbf{x}(t), t) = 0.5\mathbf{x}(t)^T\mathbf{P}_k(t)\mathbf{x}(t)$ *satisfies*

$$s_k(\mathbf{x}_k(t), u_k(t), t) = \max_{\mathbf{x} \in \mathscr{X}(t)} s_k(\mathbf{x}, u_k(t), t) \tag{10}$$

Proof.

By substitution of q_k,$\mathbf{f}$,l in Eq. (5a) and using Eq. (9). □

- The dependency of Krotov's method on a Krotov function makes it somewhat abstract. Theorems 5 and 6 turn it into a concrete solution method for the addressed CBQR problem.
- As J has a lower bound, we are assured that the sequence $\{(\mathbf{x}_k, u_k)\}$ gets arbitrary close to some admissible process $(\mathbf{x}^*, u^*)$, which is the optimal one.

Input: $\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{Q}, r, \mathbf{x}(0)$.

Initialization:

1. Define a convergence tolerance - $\epsilon > 0$.
2. Set $u_0 = 0$ and solve:

$$\begin{aligned}\dot{\mathbf{x}}_0(t) =& \mathbf{A}\mathbf{x}_0(t); \quad \mathbf{x}(0) \\ \dot{\mathbf{P}}_0(t) =& -\mathbf{P}_0(t)\mathbf{A} - \mathbf{A}^T\mathbf{P}_0(t) - \mathbf{Q}; \ \mathbf{P}_0(t_f) = \mathbf{0}\end{aligned}$$

3. Compute:

$$J_0(\mathbf{x}_0, u_0) = \frac{1}{2}\int_0^{t_f} \mathbf{x}_0(t)^T\mathbf{Q}\mathbf{x}_0(t)\mathrm{d}\,t$$

For $k = \{0, 1, 2, 3, 4, \ldots\}$:

1. Propagate to the improved process by solving:

$$\dot{\mathbf{x}}_{k+1}(t) = [\mathbf{A} - \hat{u}_{k+1}(\mathbf{x}_{k+1}(t), t)\mathbf{bc}]\mathbf{x}_{k+1}(t); \quad \mathbf{x}_{k+1}(0) = \mathbf{x}(0)$$

   where

$$\hat{u}_{k+1}(\mathbf{x}_{k+1}(t), t) = \max\left\{\frac{\mathbf{x}_{k+1}(t)^T\mathbf{P}_k(t)\mathbf{bc}\mathbf{x}_{k+1}(t)}{r}, 0\right\}$$

2. Set $u_{k+1}(t) = \hat{u}_{k+1}(\mathbf{x}_{k+1}(t), t)$.
3. Solve:

$$\begin{aligned}\dot{\mathbf{P}}_{k+1}(t) =& -\mathbf{P}_{k+1}(t)[\mathbf{A} - u_{k+1}(t)\mathbf{bc}] \\ & - [\mathbf{A}^T - u_{k+1}(t)\mathbf{c}^T\mathbf{b}^T]\mathbf{P}_{k+1}(t) - \mathbf{Q} \\ \mathbf{P}_{k+1}(t_f) =& \mathbf{0}\end{aligned}$$

4. Compute:

$$J(\mathbf{x}_{k+1}, u_{k+1}) = \frac{1}{2}\int_{0}^{t_f} \mathbf{x}_{k+1}(t)^T \mathbf{Q}\mathbf{x}_{k+1}(t) + r u_{k+1}(t)^2 \mathrm{d}\, t$$

5. If $|J(\mathbf{x}_k, u_k) - J(\mathbf{x}_{k+1}, u_{k+1})| < \epsilon$, stop iterating, otherwise - continue.

Return: $\mathbf{P}_{k+1}$.

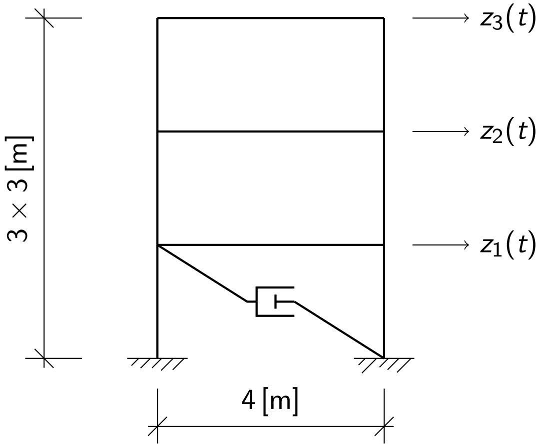
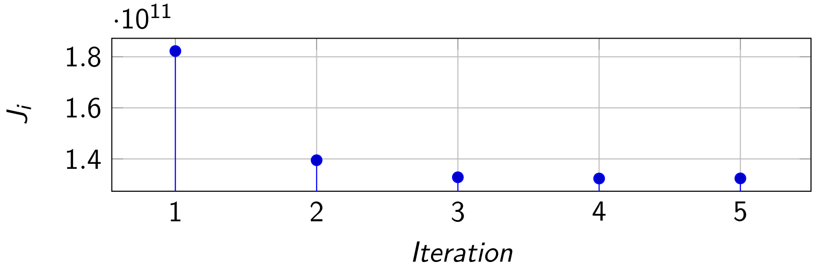
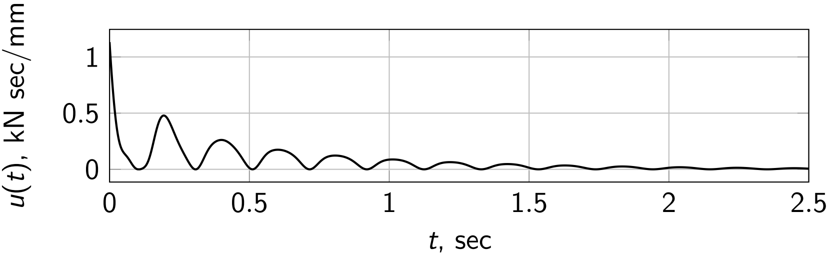


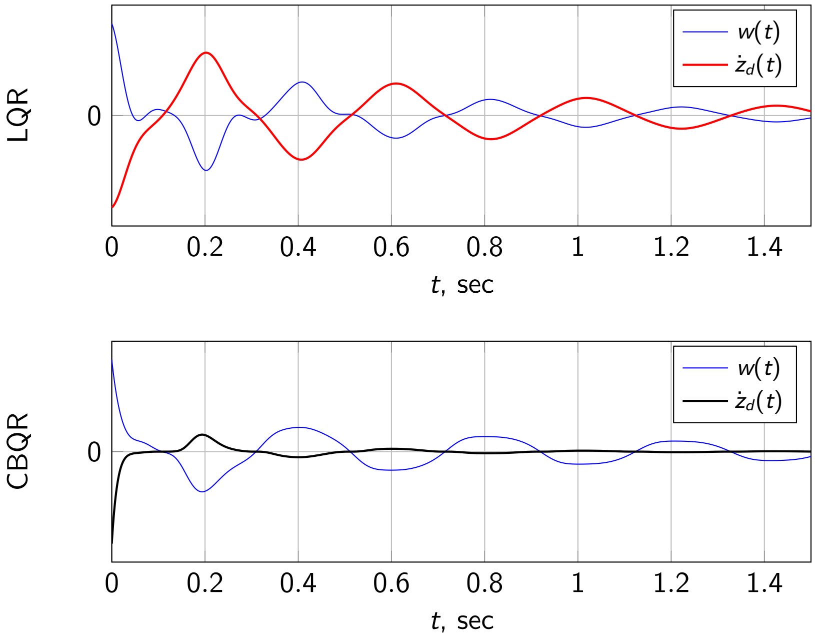
Figure: Dynamic scheme of the evaluated model.

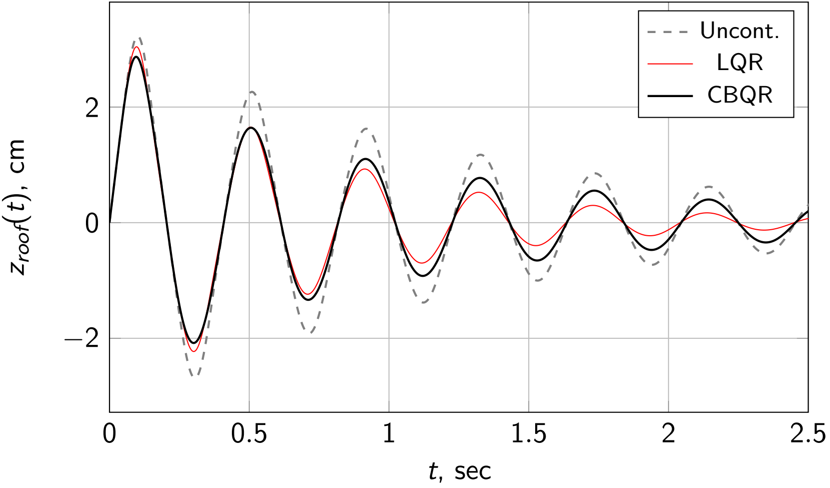


$\cdot 10^{11}$
1.8
1.6
1.4
J_i
1
2
3
4
5
Iteration



1
0.5
0
$u(t)$, kN sec/mm
0
0.5
1
1.5
2
2.5
t, sec


LQR
0
$w(t)$
$\dot{z}_d(t)$
0 0.2 0.4 0.6 0.8 1 1.2 1.4
t, sec
CBQR
0
$w(t)$
$\dot{z}_d(t)$
0 0.2 0.4 0.6 0.8 1 1.2 1.4
t, sec



Uncont.
LQR
CBQR
2
0
−2
$z_{roof}(t)$, cm
0
0.5
1
1.5
2
2.5
t, sec

- In this study a semi-active control design problem was formulated as an optimal control problem for a free vibrating bilinear system with a constrained single control signal and a quadratic performance index.
- The problem was solved by Krotov's method and a Krotov function sequence that suites the CBQR problem was found.
- The solution was organized as an algorithm, which requires the solution of the states equation and a differential Lyapunov equation in each iteration.
- The algorithm convergence is guaranteed by virtue of Krotov's method properties.
- The method efficiency is demonstrated in a numerical example.

Thank You!