Graph-Based Model Reduction of the Controlled Consensus Protocol

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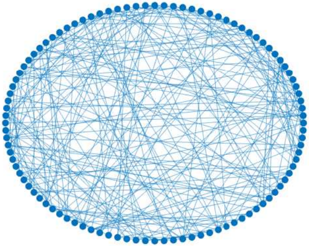
Introduction

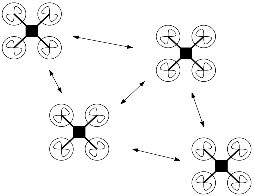
Overview

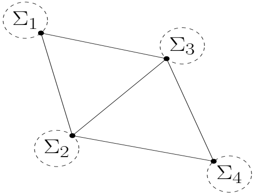
- Modern applications introduce more and more complex systems such as multi-agent systems.
- Reduced Models are required whenever it is computationally infeasible to implement, analyze or simulate the full order system.
- Some applications require reduced models to preserve structural properties of the full-order system.



Multi-Agent Systems

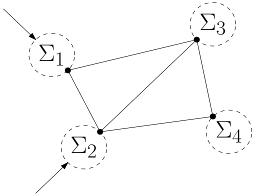
- Multi-agent systems have a unique graph-based structure where each node is an agent, and edges represent the interaction between adjacent agents.





Multi-Agent Systems

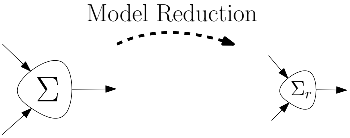
- This *multi-agent system* can be described by a *network structure* $\mathcal{M} = (\mathcal{G}, \mathcal{U})$,
  - a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$
  - a subset $\mathcal{U} \subseteq \mathcal{V}$ of agents, the leaders, are subject to external inputs

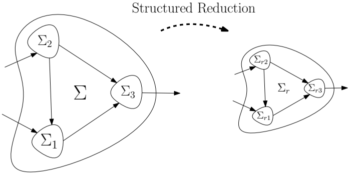


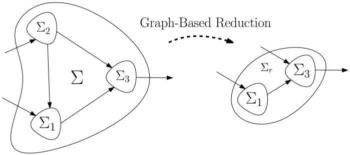
- The controlled consensus protocol is a benchmark multi-agent system and an important case study for model reduction.

Graph-Based Model Reduction

- In this study, we present a new kind of model reduction, named "graph-based model reduction" that preserves a graph structure.







The Controlled Consensus Protocol

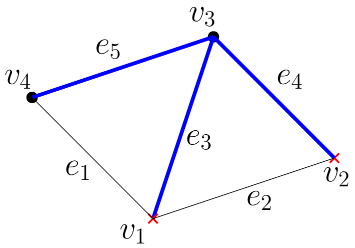
- In the controlled consensus protocol, all agents try to achieve agreement with their adjacent agents.
- The *controlled consensus* over the *network structure* $\mathcal{M} = (\mathcal{G},\mathcal{U})$ has a realization

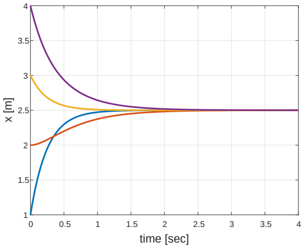
$$\Sigma_{\mathcal{M}} \begin{cases} \dot{x} & = -L\left(\mathcal{G}\right)x + B\left(\mathcal{U}\right)u \\ y & = W^{\frac{1}{2}}\left(\mathcal{G}\right)E^{T}\left(\mathcal{G}\right)x \end{cases},$$

- The matrix $L\left(\mathcal{G}\right)$ is the graph Laplacian
- The matrix $B\left(\mathcal{U}\right) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{U}|}$ maps each of the inputs to the corresponding leader's node in the network,
  - $\left[B\left(\mathcal{U}\right)\right]_{ij} = 1$ if v_i is the j'th input node and 0 otherwise.
- The matrix $W\left(\mathcal{G}\right)$ is the diagonal of the edge weights and $E\left(\mathcal{G}\right)$ is the incidence matrix.

The Controlled Consensus Protocol

- As an example, consider the consensus protocol over a connected graph $\mathcal{G}$ of order 4 with 5 unit-weight edges and two input nodes (red nodes).
- A spanning tree $\mathcal{T}(\mathcal{G})$ (blue edges) can be found.





$$L(\mathcal{G}) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad B(\mathcal{U}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E(\mathcal{G}) = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The Edge Agreement Protocol

- The *controlled consensus model* is not a minimal realization.
- Applying the transformation $x_\tau = E^T\left(\mathcal{T}\left(\mathcal{G}\right)\right)x$ leads to the so-called *edge agreement protocol* which is minimal [zelazo2011].
- We monitor the signal $z_\tau = Q^{\frac{1}{2}}\left(\mathcal{G}\right)x_\tau$ corresponding only to the outputs on $\mathcal{T}$ (i.e., we use z_τ in place of the output y).
- The *edge agreement protocol* is then

$$\hat{\Sigma}_{\mathcal{M}} \begin{cases} \dot{x}_\tau & = -L_{ess}\left(\mathcal{G}\right)x_\tau + E^T\left(\mathcal{T}\left(\mathcal{G}\right)\right)B\left(\mathcal{U}\right)u \\ z_\tau & = Q^{\frac{1}{2}}\left(\mathcal{G}\right)x_\tau \end{cases}.$$

Problem Formulation

- Construct a reduced structure $\mathcal{M}_r = (\mathcal{G}_r, \mathcal{U}_r)$ with a reduced graph $\mathcal{G}_r = (\mathcal{V}_r, \mathcal{E}_r, \mathcal{W}_r)$ of order $r < n$ and input set $\mathcal{U}_r \subseteq \mathcal{V}_r$.
- The reduced-order edge agreement protocol is then,

$$\hat{\Sigma}_{\mathcal{M}_r} \begin{cases} \dot{x}_{\tau_r} & = -L_{ess}\left(\mathcal{G}_r\right) x_{\tau_r} + E^T\left(\mathcal{T}\left(\mathcal{G}_r\right)\right) B\left(\mathcal{U}_r\right) u_r \\ z_{\tau_r} & = Q^{\frac{1}{2}}\left(\mathcal{G}_r\right) x_{\tau_r} \end{cases}.$$

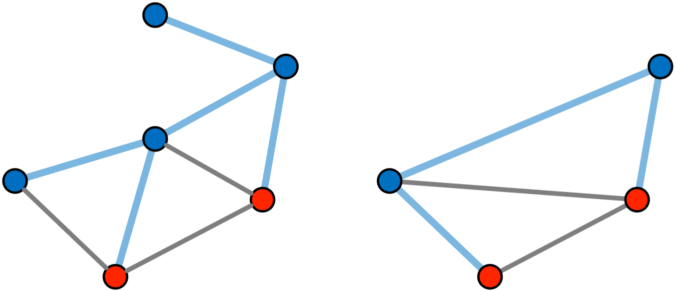
Target structure reduction

Find the reduced structure $\mathcal{M}_r^*$ that will minimize a reduction error cost

$$\min_{\mathcal{M}_r} \ \mathcal{J}\left(\hat{\Sigma}_{\mathcal{M}_r}, \hat{\Sigma}_{\mathcal{M}}\right).$$

Reduction of the Controlled Consensus Protocol

- For model reduction we need an error system.
- We use a tree to reduced-tree mapping to produce an output error signal $x_\tau - T_{(\mathcal{T},\mathcal{T}_r)} x_{\tau_r}$.
- A reduction error system with state $x_e = \left[\begin{array}{cc} x_\tau^T & x_{\tau_r}^T \end{array} \right]^T \in \mathbb{R}^{n+r-2}$ can then be constructed.



Reduction of the Controlled Consensus Protocol

- The $\mathcal{H}_2$ performance of the edge agreement protocol of order n with m input nodes is
$$\left\|\hat{\Sigma}_{\mathcal{M}}\right\|_{\mathcal{H}_2}^2 = \frac{m}{2}\left(1 - \frac{1}{n}\right).$$
- This $\mathcal{H}_2$ performance may be interpreted as the steady-state consensus dispersion for white noise input.

$\mathcal{H}_2$ reduction error cost

$$\mathcal{J}\left(\Sigma_{\mathcal{M}_r}, \Sigma_{\mathcal{M}}\right) = \frac{\left\|\Sigma_e\left(\mathcal{M}, \mathcal{M}_r\right)\right\|_{\mathcal{H}_2}^2}{\left\|\hat{\Sigma}_{\mathcal{M}}\right\|_{\mathcal{H}_2}^2}$$

Reduction by Graph Contractions

Optimal Graph-Based Model Reduction

- Solving the *target structure reduction* problem requires to find the optimal graph of order r.
- The number c_r of simple unweighted connected graphs increases exponentially [Wilf1994], e.g., for $r = 1, \ldots, 6$,

$$c_r = 1,\ 1,\ 4,\ 38,\ 728,\ 26704.$$

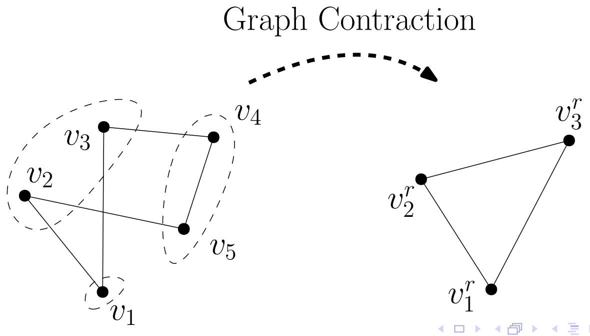
- Finding a solution to the *target structure reduction* problem may become numerically intractable for a moderate number of nodes.
- We restrict the class of graph reductions in a way that will allow us to find suboptimal constructive solutions.

Reduction by Graph Contractions

- It is expected that an optimal reduced structure $\mathcal{M}_r^*$ will have some functional dependency on the full structure $\mathcal{M}$.
- Vertex partitions have been widely used in graph theory, e.g., for graph clustering [Schaeffer2007] and in the study of network communities [Newman2004].
- Vertex partitions have been also used for constructing projection-based model reductions of multi-agent systems as the consensus protocol [Monshizadeh2014] and bidirectional networks [Ishizaki2014].

Reduction by Graph Contractions

- Here we use vertex partitions as a basis for a constructive method for performing structure reductions.
- An r-partition $\pi = \{C_1, C_2, \ldots, C_r\}$ is a set of r cells, where each cell contains a subset of nodes of the full graph.
- Each cell is contracted to a single node in the reduced graph.
- All edges connecting nodes between two cells are merged to a single edge between the corresponding nodes in the reduced graph.

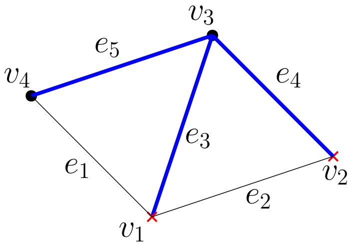


Reduction by Graph Contractions

- Finding the optimal graph contraction is combinatorially hard, and we must further restrict the class of reductions.
- We derive edge-based contractions and tree-based contractions.
- The greedy-edge tree-based contraction is suggested as an efficient suboptimal graph-reduction algorithm.

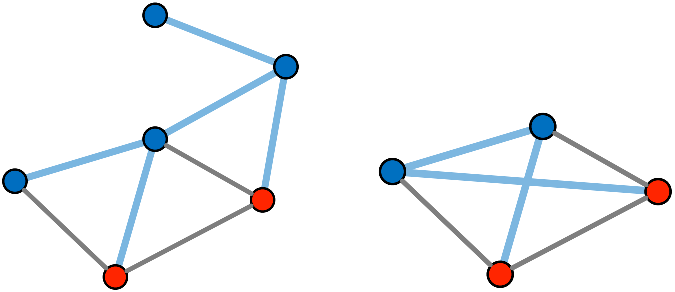
Reduction method	Required number of reductions
Graph contractions	$S(n, r) \sim \frac{r^n}{r!}$
Edge-based contractions	$\binom{\lvert\mathcal{E}(\mathcal{G})\rvert}{n-r}$
Tree-based contractions	$t(\mathcal{G}) \times \binom{n-1}{n-r}$
Greedy-edge tree-based contractions	$n-1$

1. Choose a tree $\mathcal{T}(\mathcal{G})$ in the graph.
2. Perform a separate edge-based contraction to each of the $n-1$ tree edges and calculate the reduction cost.
3. Select the $n-r$ tree edges that have minimal reduction error.
4. Perform edge-based contraction with those $n-r$ edges.



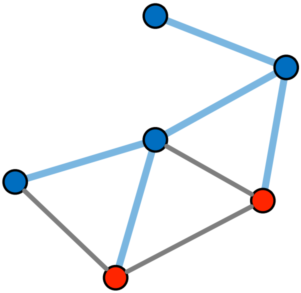
Case Study

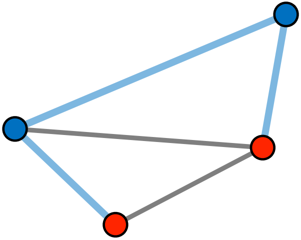
- Consider the consensus protocol over a connected graph $\mathcal{G}$ of order 6 with 8 unit-weight edges and two input nodes.
- We require the reduced system to be of order 4 with unit edge weights.
- In this case, the number of possible partitions is $S(6,4) = 65$ and we can find the optimal contraction $(\mathcal{J}_2 = 0.0711)$.



Case Study - Reduction of the Controlled Consensus Protocol

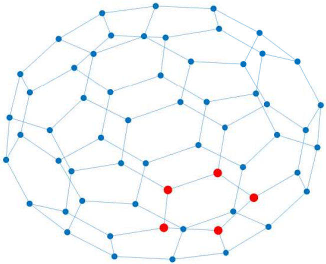
- There are $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15$ possible edge-based contractions, and we can obtain the optimal one $(\mathcal{J}_2 = 0.2266)$.
- Choosing a spanning tree (solid blue edges), we perform the greedy input-free tree-based contraction and obtain the exact same reduced graph as the optimal edge-based contraction.





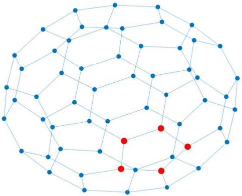
Case Study - Reduction of the Controlled Consensus Protocol

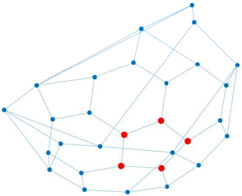
- We apply the greedy tree-based algorithm on the consensus system over a Buckminster Fuller ("Bucky") graph with 5 inputs.
- The Bucky graph is of order 60 with 90 (unit-weight) edges and we require the reduced graph to be of order 30 with unit-weight edges.



Case Study - Reduction of the Controlled Consensus Protocol

- Finding the optimal structure contraction requires the examination of $S\left(60,30\right)=9.5635\cdot 10^{53}$ contractions.
- If we restrict to edge-based contractions we have $\left(\begin{array}{c}90\\30\end{array}\right)=6.7313\cdot 10^{23}$ cases.
- A sub-optimal reduced structure is obtained with the greedy tree-based contraction, with reduction error $\mathcal{J}_2=0.0294$.





Summary and Conclusions

Summary

- We have defined the graph-based model reduction of the controlled-consensus protocol.
- The greedy tree-based contraction algorithm has been suggested as a suboptimal efficient solution and demonstrated with an $\mathcal{H}_2$ reduction error.
- Various other suboptimal structure reduction methods can be derived for the problem.
- Other reduction error metrics may be considered for the reduced consensus system.
- A similar graph-based model reduction framework can be applied to other multi-agents system as smart-grids, or second-order consensus models.
- This work will be presented at IFAC World Congress 2017 (Toulouse, France).