A Novel Approach to the Computation of Polyhedral Invariant Sets for Constrained Systems

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Overview

Introduction

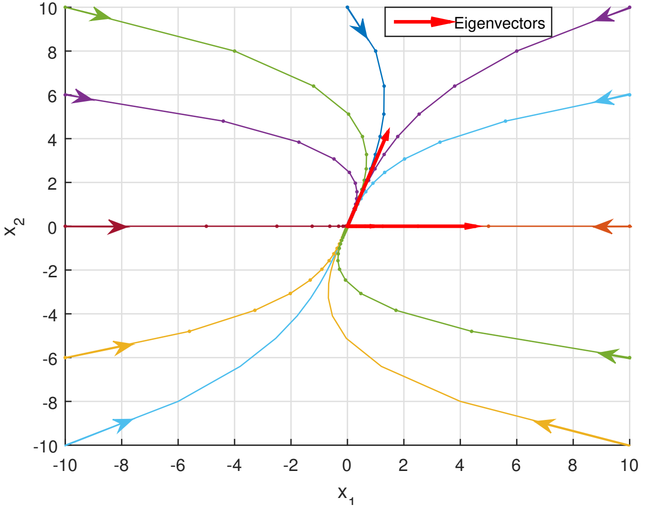
- Set invariance is a fundamental concept in both the analysis and the design of controllers for constrained linear systems, since constraints satisfaction can be guaranteed for all time if and only if the initial state is contained inside an invariant set.
- Common methods to compute maximal invariant sets may be computationally expensive, due to unbounded number of iterations.
- A new way to compute invariant sets for constrained discrete-time linear systems is proposed, with a beneficial computational burden and bounded computation time.

Basic Principle

- The method is based on auxiliary state feedback with real, distinct and stable poles assignment.
- The only assumption is that the given plant is controllable, and in the uncertain case, that it is given as a convex combination of given controllable plants.
- With the control law $u = Kx$, if the closed loop $x(k+1) = (A + BK)x(k)$ has distinct, stable, real eigenvalues, then the state trajectories will converge towards the rays defined by the eigenvectors in a non-oscillatory manner.
- Since controllability of (A, B) is assumed, a feedback controller K that gives distinct stable real eigenvalues always exists.
- Note that this controller is auxiliary only and is not related to the final control that will be designed, after the invariant set is found.

Method Basic Principle

An illustration for a second order system with eigenvalues at 0.5 and 0.8 can be seen:



Consider the controlled discrete-time linear plant without disturbances: $x(k+1) = Ax(k) + Bu(k)$, where:

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
- $x(k) \in \mathbb{R}^n$ is the state vector.
- $u(k) \in \mathbb{R}^m$ is the control input vector.

The state and control vectors are constrained to belong, respectively, to polytopic state and control constraint sets X and U of the following form:

$$\begin{cases} x(k) \in X, \ \ X = \{x \in \mathbb{R}^n : \ F_x x \leq g_x\} \\ u(k) \in U, \ \ U = \{u \in \mathbb{R}^m : \ F_u u \leq g_u\} \end{cases}$$

where the matrices F_x, F_u and the vectors g_x, g_u are assumed to be constant with $g_x > 0$, $g_u > 0$.Inequalities are taken element-wise.

Certain Plant Case

Definitions 1 and 2

The set $C \subseteq X$ is a *controlled invariant set (CIS)* for the above system if for all $x(k) \in C$, there exists a control value $u(k) \in U$ such that, $x(k+1) = Ax(k) + Bu(k) \in C$.

The set $C_{max} \subseteq X$ is the *maximal controlled invariant set* if and only if it is a controlled invariant set and contains all controlled invariant sets in X.

Definitions 3 and 4

A polyhedral set Ω is an *admissible set (AS)* for the above system for a given feedback control $u = Kx \in U$, if for all $x(k) \in \Omega$, it holds that $x(k+1) = (A + BK)x(k) \in \Omega$.
Note: an AS is a CIS.

A polyhedral set Ω_{max} is the *maximal admissible set (MAS)* if and only if it is an admissible set and contains every admissible set.

Clearly, an admissible set is a controlled invariant set.

Consider the uncertain and/or time-varying linear discrete-time plant:
$x(k+1) = A(k)x(k) + B(k)u(k)$, where:

- $A(k) \in \mathbb{R}^{n\times n}, B(k) \in \mathbb{R}^{n\times m}$ and satisfy:

$$\begin{cases} A(k) = \sum_{i=1}^{q} \beta_i(k) A_i, \quad B(k) = \sum_{i=1}^{q} \beta_i(k) B_i \\ \sum_{i=1}^{q} \beta_i(k) = 1, \qquad \beta_i(k) \geq 0 \end{cases}$$

  the matrices $A_i, B_i,\ i = 1, 2, ..., q$ are the extreme realizations of $A(k)$ and $B(k)$. $\beta_i(k)$ are unknown.
- $x(k) \in \mathbb{R}^n$ is the state vector.
- $u(k) \in \mathbb{R}^m$ is the control input vector.

As in the certain case, the state and the control are subject to polytopic constraints.

Definitions 5 and 6

The set $C \subseteq X$ is a *robust controlled invariant set* for the above system if for all $x(k) \in C$, there exists a control value $u(k) \in U$ such that, $x(k+1) = A(k)x(k) + B(k)u(k) \in C$.
The set $C_{max} \subseteq X$ is the *maximal robust controlled invariant set* if and only if it is a robust controlled invariant set and contains all controlled invariant sets in X.

Definitions 7 and 8

A polyhedral set Ω is a *robust admissible set (RAS)* for the above system for a given feedback control $u = Kx \in U$, if for all $x(k) \in \Omega$, it holds that $x(k+1) = (A(k) + B(k)K)x(k) \in \Omega$.
A polyhedral set Ω_{max} is the *maximal robust admissible set (MRAS)* if and only if it is a robust admissible set and contains every robust admissible set.

Clearly, a robust admissible set is a robust controlled invariant set.

1. For a controllable plant with the state and control constraints as described, assign distinct stable real eigenvalues λ_i , $i = 1, .., n$ for the closed-loop system by choosing K in the control law

$$u = Kx, K \in \mathbb{R}^{m \times n}$$

2. Calculate the corresponding eigenvectors, v_i , $i = 1, .., n$ and define the matrix V that contains the eigenvectors as columns,

$$V = [v_1 \;\; v_2 \;\; \ldots \;\; v_n] \in \mathbb{R}^{n \times n}$$

3. Construct a CIS, called P, as a convex combination of ray segments of maximal lenghts along the eigenvectors, $[-\alpha_i, \alpha_i]v_i$, by solving the following LP problem with decision variables $\alpha_i, i = 1, ..., n$

$$\begin{cases} \max\limits_{\alpha_i} \quad \sum\limits_{i=1}^{n} \alpha_i \qquad \alpha_i \geq 0 \\ s.t. \quad F_x\left(\alpha_i \cdot v_i\right) \leq g_x, \qquad \forall i = 1, ..., n \\ \qquad\;\; F_u\left(\alpha_i \cdot K \cdot v_i\right) \leq g_u, \;\; \forall i = 1, ..., n \end{cases}$$

Define the matrix

$$\Psi = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$P = \mathrm{Conv}\left\{\left[\mathrm{V} \cdot \Psi \quad -\mathrm{V} \cdot \Psi\right]\right\}$$

4. Let b_i denote the binary representation of the integer i, $i = 0, ..., 2^n - 1$, with each zero replaced by -1. Let $b_i(j)$ be the j'th integer of b_i. Let $\hat{p}_i = \sum_{j=1}^{n} b_i(j)\alpha_j v_j$ define the polytope $\hat{P}$ which is the convex hull of the vertices $\hat{p}_0, \ ..., \ \hat{p}_{2^n-1}$.
5. Solve the following LP problem to obtain another CIS, $\gamma\hat{P}$,

$$\begin{cases} \max\limits_{\gamma} \quad \gamma \qquad 0 \le \gamma \le 1 \\ s.t. \quad F_x\left(\gamma \cdot \hat{p}_i\right) \le g_x, \qquad \forall i = 0, ..., 2^n - 1 \\ \qquad F_u\left(\gamma \cdot K \cdot \hat{p}_i\right) \le g_u, \ \forall i = 0, ..., 2^n - 1 \end{cases}$$

6. Construct an enlarged CIS, P_f, by the convex hull operation,

$$P_f = \mathrm{Conv}\left\{\mathrm{P}, \ \gamma\hat{\mathrm{P}}\right\}$$

1. Consider a controllable plant with the state and control constraints as was described. For an arbitrary extreme plant instance (A_1, B_1) assign n distinct stable real eigenvalues for the closed-loop system by choosing K in the control law

$$u = K_1 x, K_1 \in \mathbb{R}^{m \times n}$$

2. Calculate the corresponding eigenvectors, v_i , $i = 1, .., n$ and define the matrix V that contains the eigenvectors as columns,

$$V = [v_1 \;\; v_2 \;\; ... \;\; v_n] \in \mathbb{R}^{n \times n}$$

3. Define the real-valued decision matrices $\Lambda_j, j = 2, ..., q$, of the form

$$\Lambda_j = \begin{bmatrix} \lambda_{j,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{j,n} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

   and the real-valued decision matrices, $K_j \in \mathbb{R}^{m \times n}, j = 2, .., q$.
4. Consider the remaining extreme plant instances, $A_j, B_j, j = 2, ..., q$, for each of which the closed-loop eigenvector assignment problem is now solved e.g. by LP, such that the set of eigenvectors equal to the columns of V.

$$\begin{cases} \min\limits_{K_j, \Lambda_j} \; 1 \\ s.t. \quad (A_j + B_j \cdot K_j)\, V = V \cdot \Lambda_j \\ \qquad -1 \leq \lambda_{j,i} \leq 1, \quad \forall i = 1, ..., n \end{cases}$$

- If complex Λ_j had been admissible, then the LP-problems above would have had solutions, since the eigenvector assignment problem with arbitrary eigenvalues always has a solution. Here, however, it is not guaranteed that a solution exists for all j, but, since it is an LP problem, it is decidable in polynomial time.
- An expansion to the certain plant algorithm can be made to adapt it for the parametric uncertain plant case. The rest of the details can be found in the paper.
- Another extension is to treat each vertex, which is an eigenvector candidate, separately for all extreme plant cases, in point 4 on the previous slide.

Consider $x(k+1) = Ax(k) + Bu(k)$, with

$$A = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the constraints,

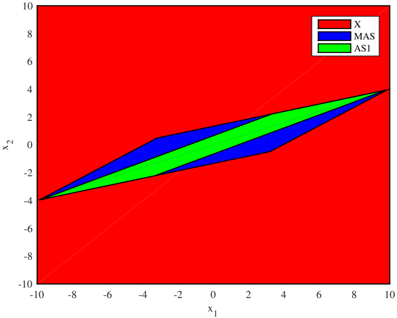
$$\begin{cases} X = \{x : \ |x_1| \leq 10, \quad |x_2| \leq 10\} \\ U = \{u : \ |u| \leq 5\} \end{cases}$$

The auxiliary state feedback controller is: K $=$ [1.09 -3.984]

Certain Plant Case

Numerical Example

Proposed algorithm computation time: 0.049 sec.
Standard algorithm computation time: 0.095 sec.



X - State constraints polytope, MAS - Corresponding maximal admissible set using the standard iterative algorithm [see refs.], AS1 - Polyhedral admissible set computed with the algorithm.

Parametric Uncertain Plant Case

Numerical Example

Consider $x(k+1) = A(k)x(k) + B(k)u(k)$ The polytopic uncertainty is defined by the extreme realizations $(A_1, B_1), (A_2, B_2)$

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

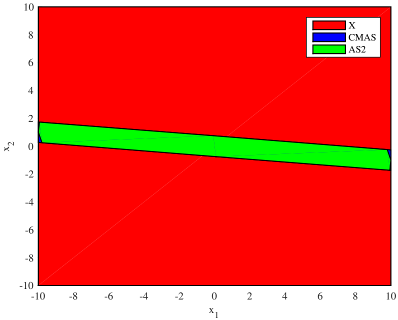
$$A_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

and the constraints,

$$X = \{x : \ |x_1| \leq 10, \quad |x_2| \leq 10\}$$

$$U = \{u : \ |u| \leq 1\}$$

The auxiliary state feedback controllers are: $K_1 = [-0.1 \ \ -1.01]$
$K_2 = [-0.1333 \ \ -1.3467]$



X - State constraints polytope, CMAS - Convex hull of maximal admissible sets using standard iterative algorithm [see refs.] for each of the controllers, AS2 - Polyhedral robust controlled invariant set computed with the new algorithm.

Conclusions

- In contrast to the standard methods, the new method is not iterative, and the number of vertices of the resulting polytope is a-priori known, bounded function of the plant order.
- In contrast to the standard methods the computational times for our algorithm may vary between different runs depending on the specific choice of eigenvalues.
- While the standard algorithm may take an unbearably long time to reach a result for high-order systems our algorithm can give a result much faster.

Summary

- We considered the problem of computing invariant sets for constrained discrete-time certain and uncertain linear systems.
- We proposed a novel approach which is based on the use of auxiliary state feedback laws, that give distinct, stable and real-valued closed loop eigenvalues.
- Numerical examples were shown to demonstrate the use of the algorithm that was introduced.

References

S. Sheer and P.O. Gutman, *A novel approach to the computation of polyhedral invariant sets for constrained systems*, IEEE Conference on Computer Aided Control System Design (CACSD) 2016.

Questions?