Analysis of stability transitions in a microswimmer with superparamagnetic links

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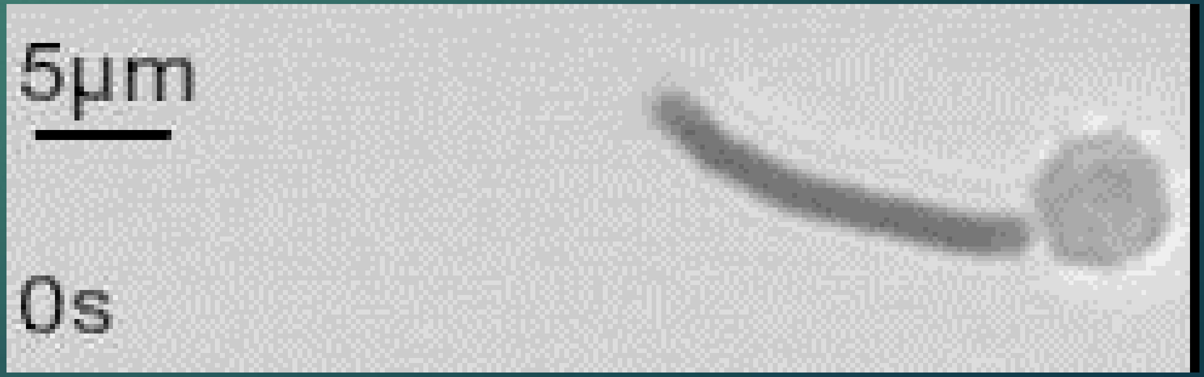
Introduction

- Dreyfus (2005): Introduced a swimmer actuated by an external magnetic field of the form:

$$\begin{pmatrix} 1 \\ \beta \sin(\omega t) \end{pmatrix} B_x$$



Dreyfus (Nature 2005) – supplementary video 2

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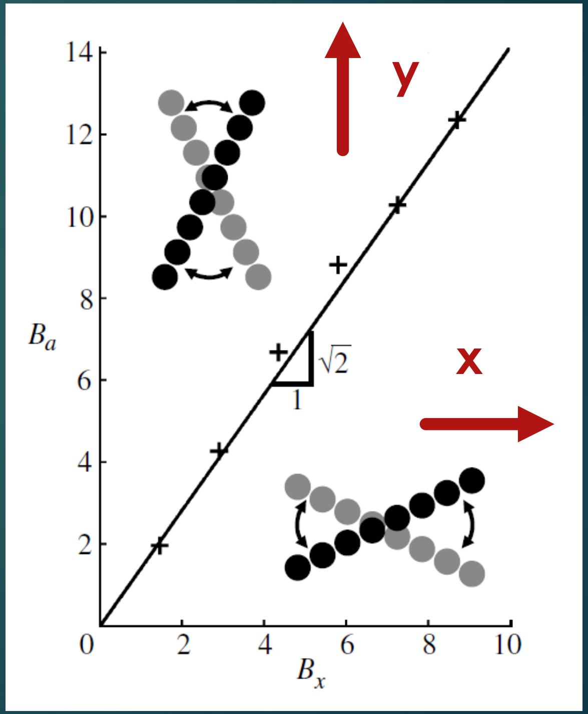

Roper et al (PRSA 2008)

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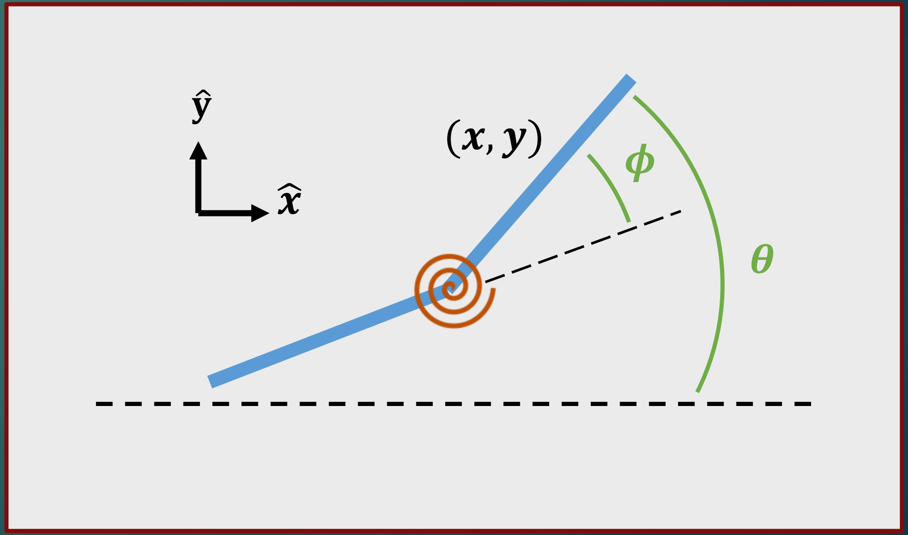
- **Our goal: analyzing the swimmer using theoretical models**

Microswimmer modelling

- Low Reynolds number hydrodynamics
- No inertia – quasi-static motion

$$\sum F_i = 0\,, \sum M_i = 0$$

- 3 physical mechanisms:
  - Elasticity
  - Magnetic torque
  - Viscous drag



Robotic microswimmer modelling – magnetic forces

- Magnetic moment – relates the field to the torque:

$$\boldsymbol{L} = \boldsymbol{\mathcal{M}} \times \boldsymbol{B}$$

- Ferromagnetic materials – constant magnetic moment:

$$\boldsymbol{\mathcal{M}} = \boldsymbol{constant}$$

- Paramagnetic materials – induced magnetic moment:

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\chi} \cdot \boldsymbol{B}$$

Robotic microswimmer modelling – calculating drag: RFT

- RFT – Resistive Force Theory:

$$F_t = -c_t \cdot v_t \,,\; F_n = -c_n \cdot v_n \,,\; M = -c_m \omega$$

- For slender link:

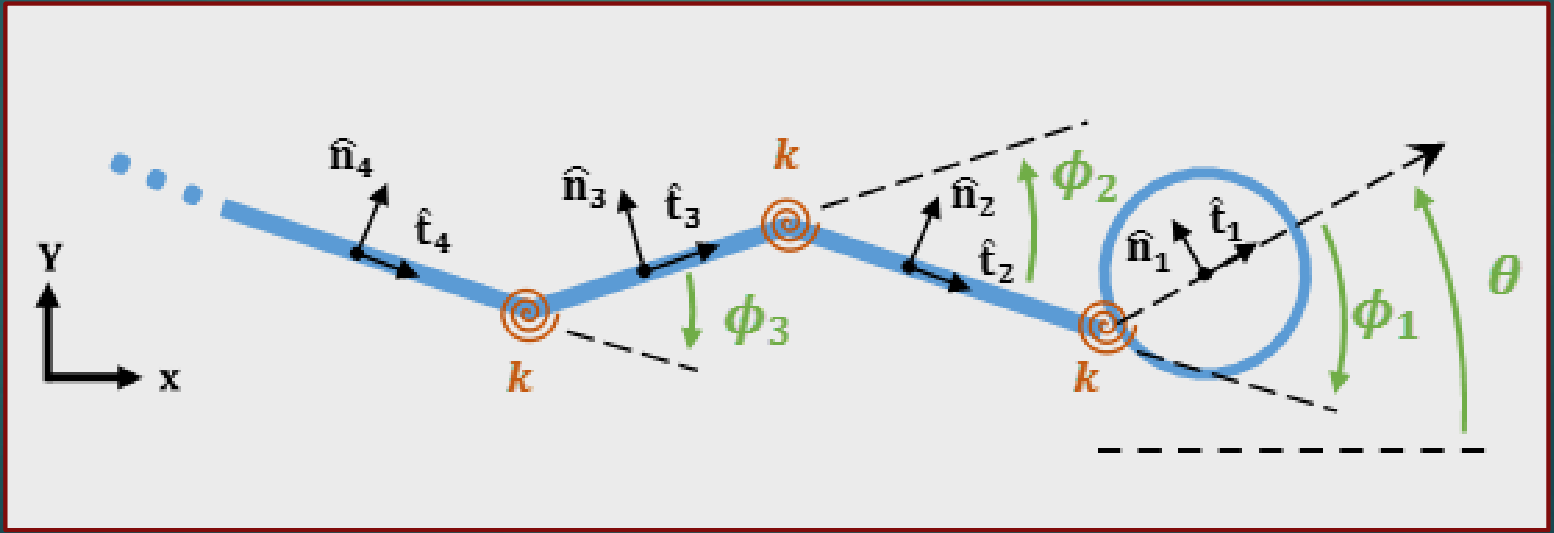
$$c_n \approx 2c_t = \frac{4\pi\mu l}{\ln\left(\frac{l}{a}\right)}, c_m = \frac{c_n l^2}{12}$$

- For spherical head:

$$c_t = c_n = 6\pi\mu r, c_m = 8\pi\mu r^3$$

Numerical analysis of multilink model

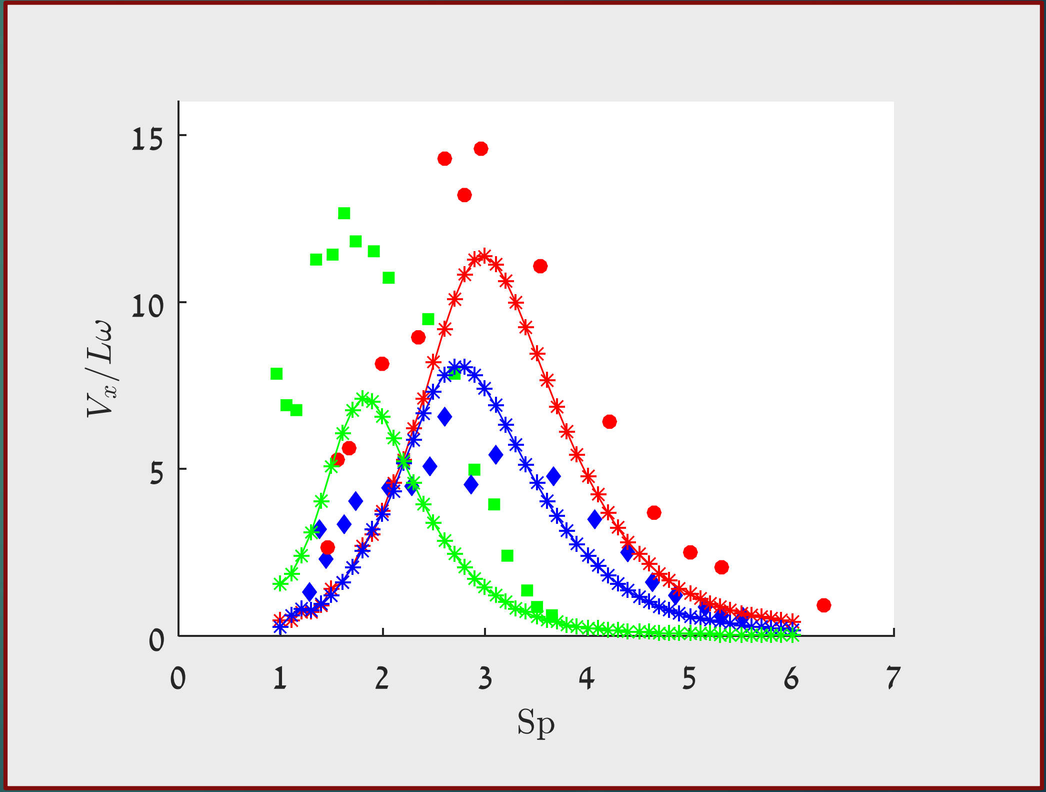
- Spherical link with a tail that consists of a chain of slender, superparamagnetic links, connected by torsion springs.
- Hydrodynamic and magnetic interactions are neglected
- External magnetic field: $\boldsymbol{B} = \begin{pmatrix} 1 \\ \beta \sin(\omega t) \end{pmatrix} B_x$



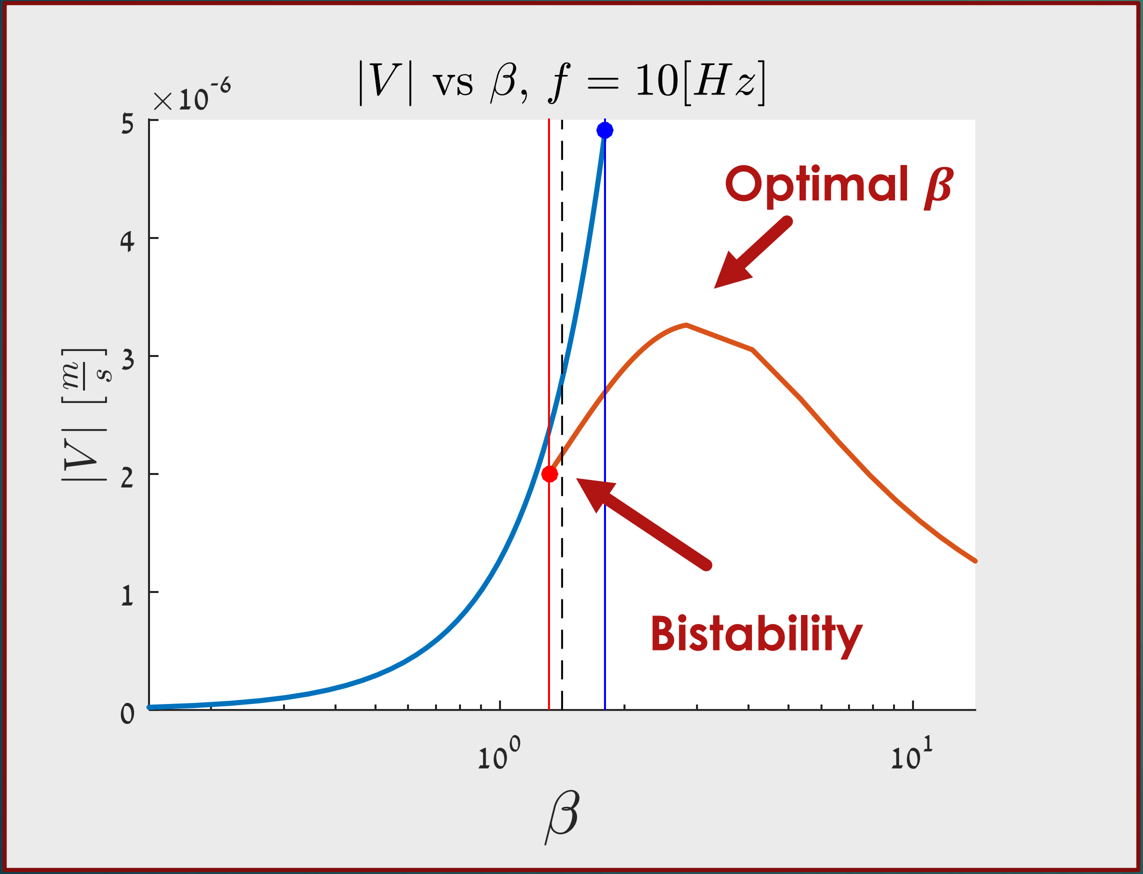
Comparison to experiments

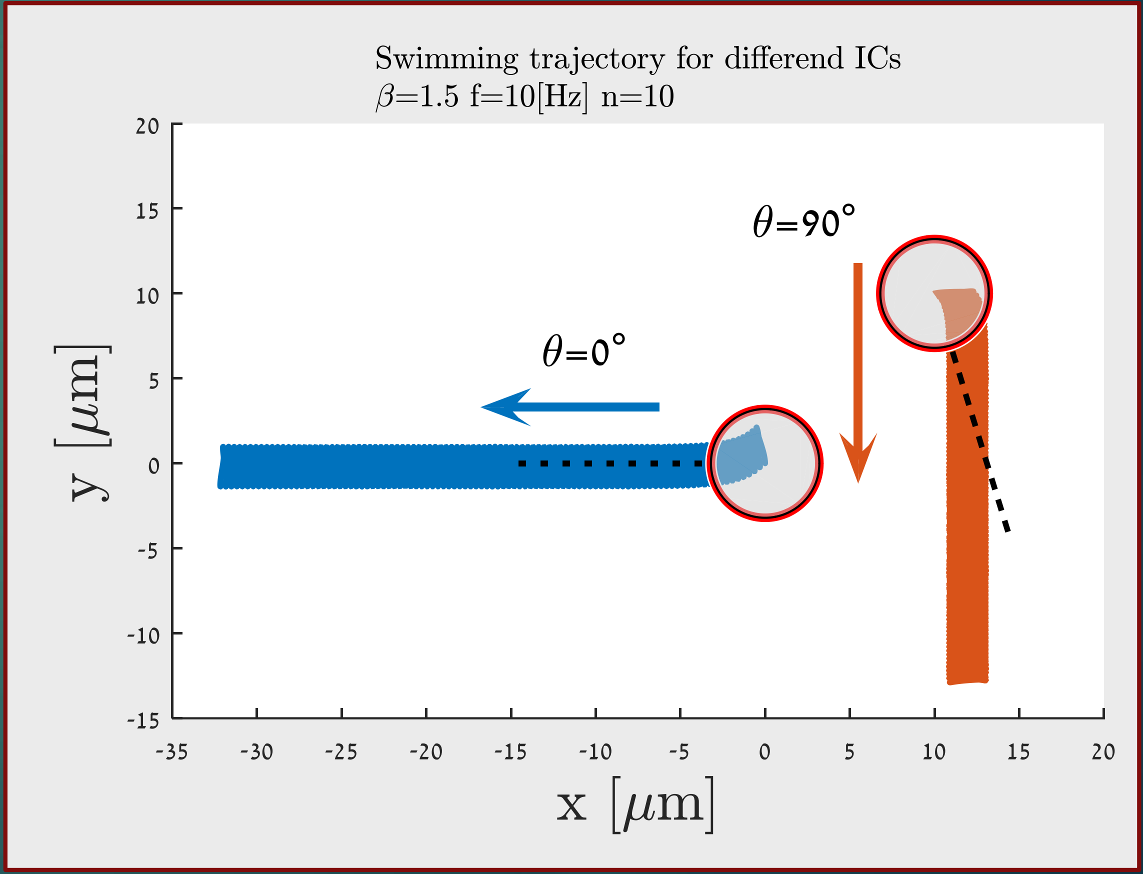
- Comparison between our numerical results and the experimental results of Dreyfus et al.
- $Sp = \frac{L}{\left(\frac{\kappa}{c_n \omega}\right)^{\frac{1}{4}}}$

Speed vs. frequency



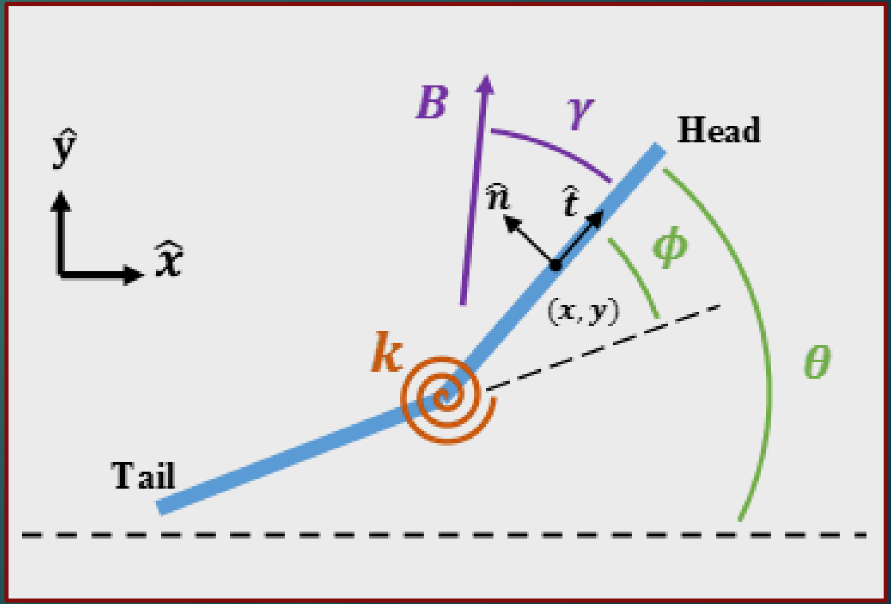
Further investigation: Bi-stability and optimal β





Analysis of a 2 link model

- We base our model on the (ferromagnetic) model introduced by Gutman and Or (2014), that did not exhibit stability transitions
- Two slender links, one **paramagnetic**, one non-magnetic, connected by a torsion spring
- Drag forces calculated using RFT (slender links)
- Torsion spring: $\tau = -k\phi$
- External magnetic field: $\boldsymbol{B} = \begin{pmatrix} 1 \\ \beta \sin(\omega t) \end{pmatrix} B_x$
- The magnetic torque generated: $\boldsymbol{L} = \Delta\chi V(\boldsymbol{t} \times \boldsymbol{B})(\boldsymbol{t} \cdot \boldsymbol{B}) \sim \sin(2\gamma)$
  - (For ferromagnetic link $\boldsymbol{L} \sim \sin(\gamma)$)



Stability of a single magnetic link in a constant field

- Ferromagnetic link: $L \sim \sin(\gamma)$
  - 2 equilibrium states

$B = const$

$stable, \gamma = 0$

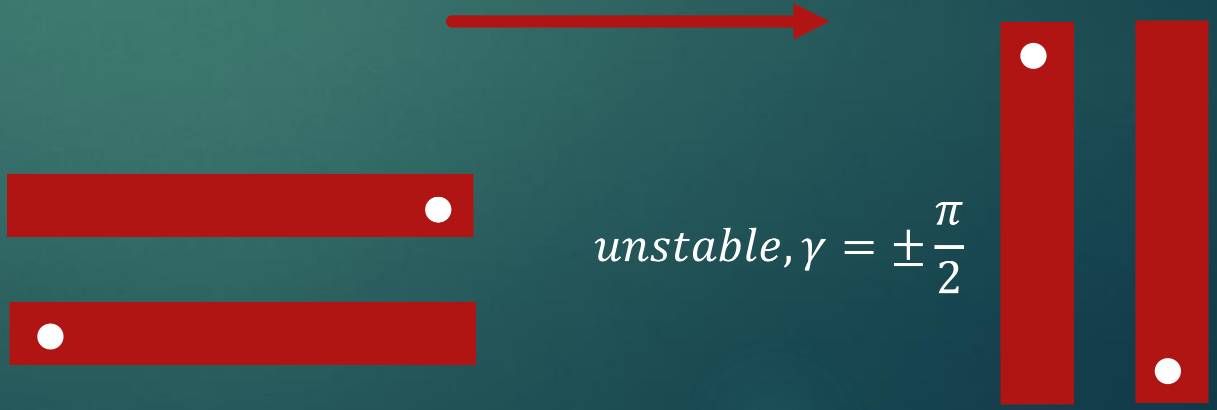
$unstable, \gamma = \pi$

- Paramagnetic link: $L \sim \sin(2\gamma)$
  - 4 equilibrium states

$B = const$

$stable, \gamma = 0, \pi$

$unstable, \gamma = \pm\frac{\pi}{2}$



Analysis of a 2 link model – Continued

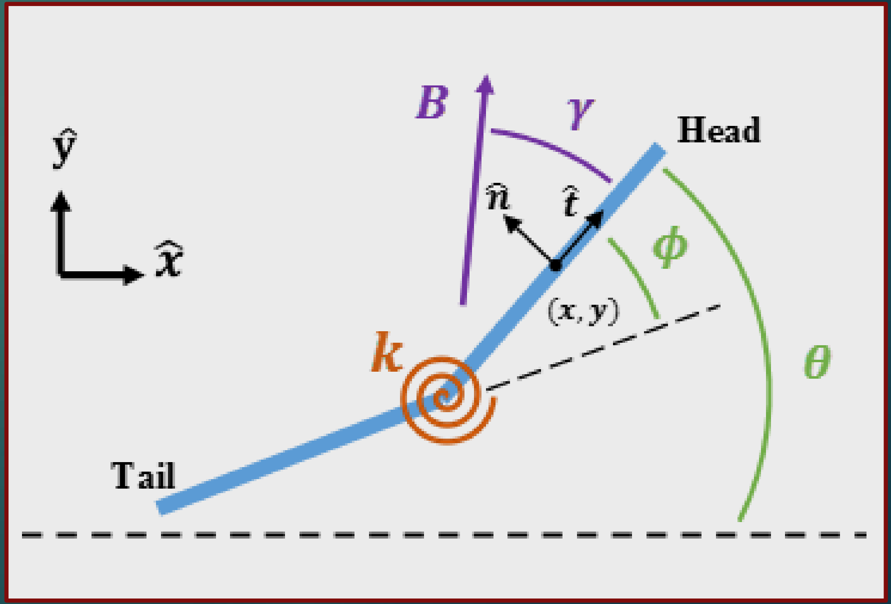
- 3 characteristic time scales:

$$t_\omega = \frac{1}{\omega} = actuation,\ \ t_m = \frac{c_t l^3}{6\Delta\chi B_x^2 v} = \frac{viscocity}{magnetic},\ \ t_k = \frac{c_t l^3}{12k} = \frac{viscosity}{elasticity}$$

  - we also denote $\alpha = \frac{t_m}{t_k} = \frac{magnetic}{elasticity}$

- Infinite domain dictates that the dynamics are independent of x, y

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(\theta,\phi) \\ \dot{\theta} = \dfrac{\phi(\cos(\phi)+3)^2}{2(\cos(2\phi)-17)}\dfrac{1}{t_k} - \dfrac{(\cos(2\phi)+19)\left(\sin(2\theta)\left(\beta^2\sin^2(\omega t)-1\right)+2\beta\cos(2\theta)\sin(\omega t)\right)}{4(\cos(2\phi)-17)}\dfrac{1}{t_m} \\ \dot{\phi} = \dfrac{(\cos(\phi)+3)^2\left((\sin(2\theta)-\beta\sin(\omega t)(2\cos(2\theta)+\beta\sin(2\theta)\sin(\omega t))\right)}{2(\cos(2\phi)-17)}\dfrac{1}{t_m} + \dfrac{(\cos(\phi)+3)^2\phi}{(\cos(2\phi)-17)}\dfrac{1}{t_k} \end{cases}$$



Fast actuation and soft swimmer – Method of Multiple scales

- Taking the ratios $\frac{t_\omega}{t_m}, \frac{t_\omega}{t_k} \approx O(\epsilon) \ll 1$
- Using the method of multiple scales
  - Introducing fast and slow time scales: $T_0 = \frac{t}{t_m}, T_1 = \epsilon t$
  - Expanding the solutions
    $$\boldsymbol{q} = \boldsymbol{q_0}(T_0, T_1) + \epsilon \boldsymbol{q_1}(T_0, T_1) + \epsilon^2 \boldsymbol{q_2}(T_0, T_1) + \cdots = \boldsymbol{q_0} + \boldsymbol{\Delta q}$$
  - Equating coefficients of ϵ
  - Requiring elimination of secular terms
- 0^{th} order is only slow dynamics:
  $$\theta_0 = \Theta_0(T_1), \phi_0 = \Phi_0(T_1)$$

Fast actuation and soft swimmer–continued

- Slow dynamics equations obtained from eliminating secular terms
- Equilibrium points at $\Theta_e = \left\{0, \frac{\pi}{2}\right\}, \Phi_e = 0$

$$D_1\Theta_0 = \frac{d\Theta_0}{dT_1} = \frac{4\alpha\Phi_0(\cos(\Phi_0)+3)^2 - \left(\beta^2 - 2\right)\sin(2\Theta_0)(\cos(2\Phi_0)+19)}{8(\cos(2\Phi_0)-17)}$$

$$D_1\Phi_0 = \frac{d\Phi_0}{dT_1} = \frac{(\cos(\Phi_0)+3)\left(4\alpha\Phi_0 - \left(\beta^2 - 2\right)\sin(2\Theta_0)\right)}{8(\cos(\Phi_0)-3)}$$

Fast actuation and soft swimmer– continued

- Slow dynamics equations obtained from eliminating secular terms
- Equilibrium points at $\Theta_e = \left\{0, \frac{\pi}{2}\right\}, \Phi_e = 0$
- Linearization about both equilibrium points yields:

Linearization about $\Theta_e = 0$

$$\begin{pmatrix} \dot{\Theta}_0 \\ \dot{\Phi}_0 \end{pmatrix} = \begin{pmatrix} \frac{5}{16}(\beta^2 - 2) & -\frac{1}{2}\alpha \\ \frac{1}{2}(\beta^2 - 2) & -\alpha \end{pmatrix} \begin{pmatrix} \Theta_0 \\ \Phi_0 \end{pmatrix}$$

Stable for $\beta < \sqrt{2}$

Unstable for $\beta > \sqrt{2}$

Corresponds to $V_x \neq 0, V_y = 0$ (not shown)

Linearization about $\Theta_e = \frac{\pi}{2}$

$$\begin{pmatrix} \dot{\Theta}_0 \\ \dot{\Phi}_0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{16}(\beta^2 - 2) & -\frac{1}{2}\alpha \\ -\frac{1}{2}(\beta^2 - 2) & -\alpha \end{pmatrix} \begin{pmatrix} \Theta_0 \\ \Phi_0 \end{pmatrix}$$

Unstable for $\beta < \sqrt{2}$

Stable for $\beta > \sqrt{2}$

Corresponds to $V_x = 0, V_y \neq 0$ (not shown)

Stability transition for $\beta \to \sqrt{2}$, with no dependence on ω is confirmed

Optimal β for velocity in period is also found (not shown)

Bistability regions are not observed

Fast actuation and **stiff** swimmer – perturbation expansion

- Taking the ratios $\frac{t_\omega}{t_m} = O(\epsilon) \ll 1\,, \frac{t_\omega}{t_k} = O(1)$

$$\dot{\theta} = \frac{\phi(\cos(\phi)+3)^2}{2(\cos(2\phi)-17)}\frac{1}{t_k} - \frac{(\cos(2\phi)+19)\left(\sin(2\theta)\left(\beta^2\sin^2(\omega t)-1\right)+2\beta\cos(2\theta)\sin(\omega t)\right)}{4(\cos(2\phi)-17)}\frac{1}{t_m}$$

$$\dot{\phi} = \frac{(\cos(\phi)+3)^2\left((\sin(2\theta)-\beta\sin(\omega t)(2\cos(2\theta)+\beta\sin(2\theta)\sin(\omega t)))\right)}{2(\cos(2\phi)-17)}\frac{1}{t_m} + \frac{(\cos(\phi)+3)^2\phi}{(\cos(2\phi)-17)}\frac{1}{t_k}$$

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- Substituting $\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$ and equating coefficients of ϵ

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- 1st order approximation yields a set of equations linear in ϕ

$$\dot{\theta} = -\frac{1}{2}\alpha\phi - \frac{5}{16}\left(1-\beta^2\sin^2(\omega t)\right)\sin(2\theta) + \frac{5}{8}\beta\sin(\omega t)\cos(2\theta)$$

$$\dot{\phi} = -\phi\alpha - \frac{1}{2}\left(1-\beta^2\sin^2(\omega t)\right)\sin(2\theta) + \beta\sin(\omega t)\cos(2\theta)$$

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- 1st order approximation yields a set of equations linear in ϕ
- Re-writing the system as a 2nd order ODE in θ only
  - periodic solution $\theta_p(t)$ oscillating about $\theta_e = \left\{0, \frac{\pi}{2}\right\}$

$$\ddot{\theta} + \left(\alpha + \frac{5}{8}\cos(2\theta)\left(1-\beta^2\sin^2(\omega t)\right) + \frac{5}{4}\beta\sin(2\theta)\sin(t\omega)\right)\dot{\theta} + \frac{1}{16}\left(\alpha\left(1-\beta^2\sin^2(\omega t)\right) - 10\beta^2\omega\sin(2t\omega)\right)\sin(2\theta) = \frac{1}{8}\beta(\alpha\sin(t\omega) + 5\omega\cos(t\omega))\cos(2\theta)$$

Fast actuation and **stiff** swimmer – perturbation expansion

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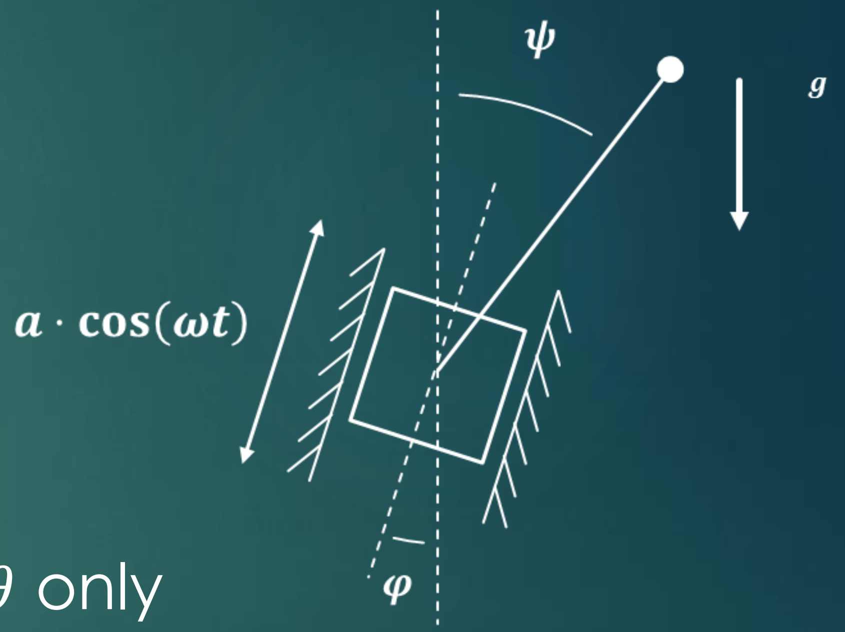
- Inclined Kapitza pendulum: $\ddot{\psi} - \left(\frac{g}{l} + \frac{a\omega^2}{l}\cos(\varphi)\cos(\omega t)\right)\sin(\psi) = -\frac{a\omega^2}{l}\sin(\varphi)\cos(\omega t)\cos(\psi)$



Variational equation

- Assuming a solution of the form

$$\theta(t) = \theta_p(t) + \delta(t)$$

- Substitute solution form into nonlinear equation
- Expand the equation about $\delta = 0$
- A linear Hill equation in δ is obtained:

$$\ddot{\delta} + p_1(\theta_p, t)\dot{\delta} + p_2(\theta_p, t)\delta = 0$$

Approximation of θ_p

- Linearizing the 2nd order ODE in θ about $\theta_e = 0, \frac{\pi}{2}$ yields a Hill equation of the form

$$\ddot{\tilde{\theta}} + \left(A_1 + 2B_1\cos(2t\omega)\right)\dot{\tilde{\theta}} + \left(A_2 + 2B_2\cos(2t\omega) + 2\omega C_2\sin(2t\omega)\right)\tilde{\theta} = f(\alpha, \beta, \omega, t)$$

$$\textit{where} \quad \tilde{\theta} = \theta - \theta_e$$

- Using harmonic balance, an approximation of the periodic solution is obtained:
  - $\tilde{\theta} \approx \tilde{\theta}_K = \sum_{k=1}^{K} a_k \cos(k\omega t) + b_k \sin(k\omega t)$
- $\theta_p(t) \approx \theta_e + \tilde{\theta}_K(t)$

$$A_1 = \alpha + \frac{5}{16}(\beta^2 - 2)\cos(2\theta_e), A_2 = \frac{1}{16}\alpha(\beta^2 - 2)\cos(2\theta_e)$$

$$B_1 = -\frac{5}{32}\beta^2\cos(2\theta_e), B_2 = -\frac{1}{32}\alpha\beta^2\cos(2\theta_e)$$

$$C_2 = \frac{5}{16}\beta^2\cos(2\theta_e)$$

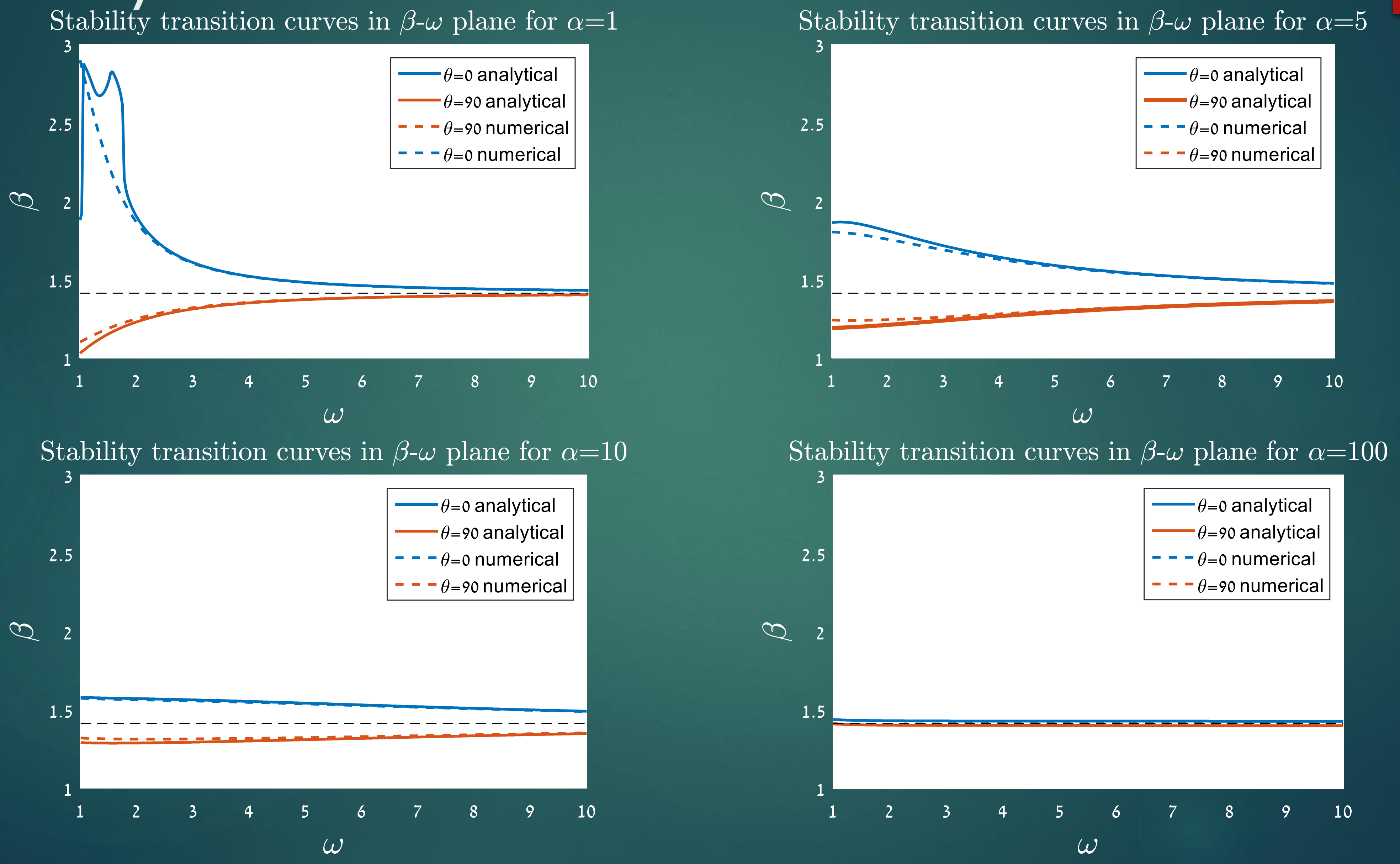
Hill's determinant method

- Expanding the coefficients of $\delta, \dot{\delta}$ into a Fourier series yields a Hill equation

  $\ddot{\delta} + p_1(t)\dot{\delta} + p_2(t)\delta = 0$, where p_1, p_2 periodic, with period $T = \pi/\omega$
- Solutions corresponding to stability transitions have a period of $\frac{2\pi}{\omega}$ (Floquet theory)
- Substituting $\delta = M_0 + \sum_{k=1}^{K} M_k \cos(n\omega t) + N_k \sin(n\omega t)$
- Equating coefficients of each harmonic
- Obtaining a homogenous, algebraic system $\boldsymbol{Hx} = 0$
- We require that $\det(H) = 0$
- The solutions of $\mathbf{det}(\boldsymbol{H}) = \mathbf{0}$ are the stability transition curves

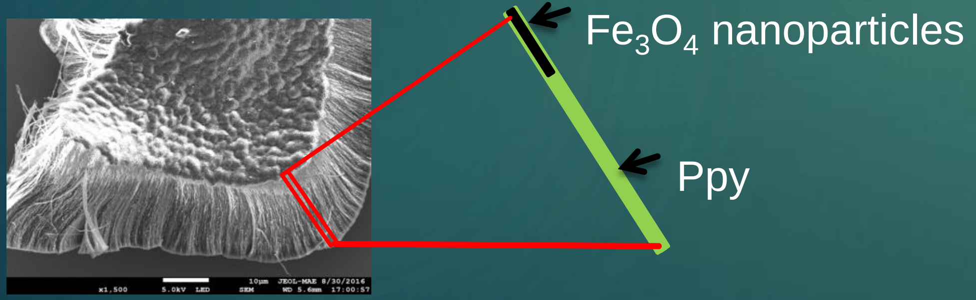
$$H(\alpha,\beta,\omega) = \begin{pmatrix} H_{11} & 0 & 0 & H_{14} & H_{15} \\ 0 & H_{22} & H_{23} & 0 & 0 \\ 0 & H_{32} & H_{33} & 0 & 0 \\ H_{41} & 0 & 0 & H_{44} & H_{45} \\ H_{51} & 0 & 0 & H_{54} & H_{55} \end{pmatrix}$$

Analytical vs Numerical

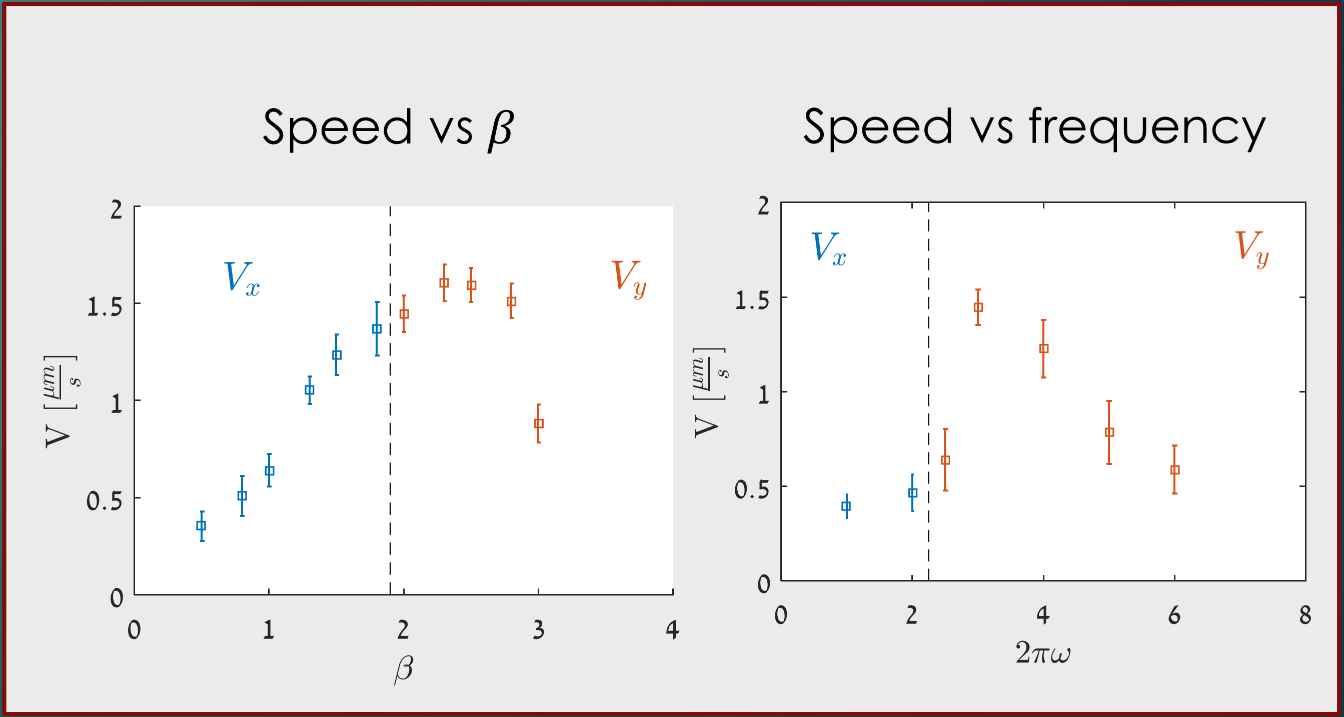


Zhang and Jin experiments

- Experiments conducted by the research group of Professor Zhang from the Chinese University of Hong Kong
- Swimmer fabricated out of Ppy elastic tail embedded with paramagnetic paricles



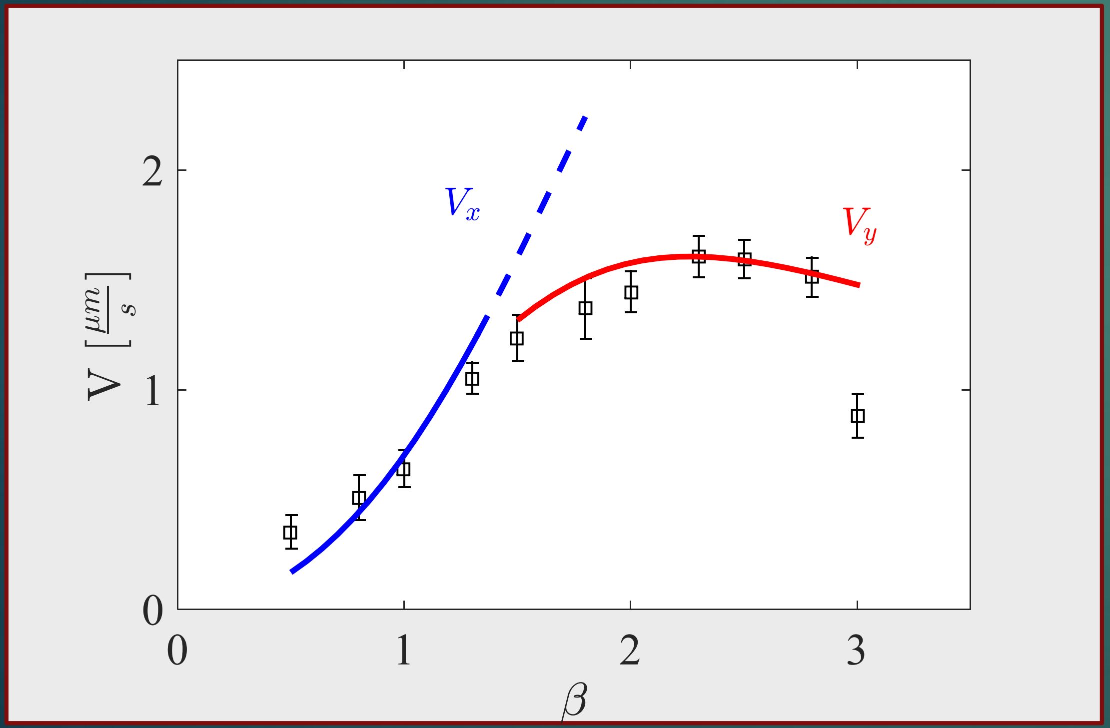
SEM image of the as-prepared nanowires



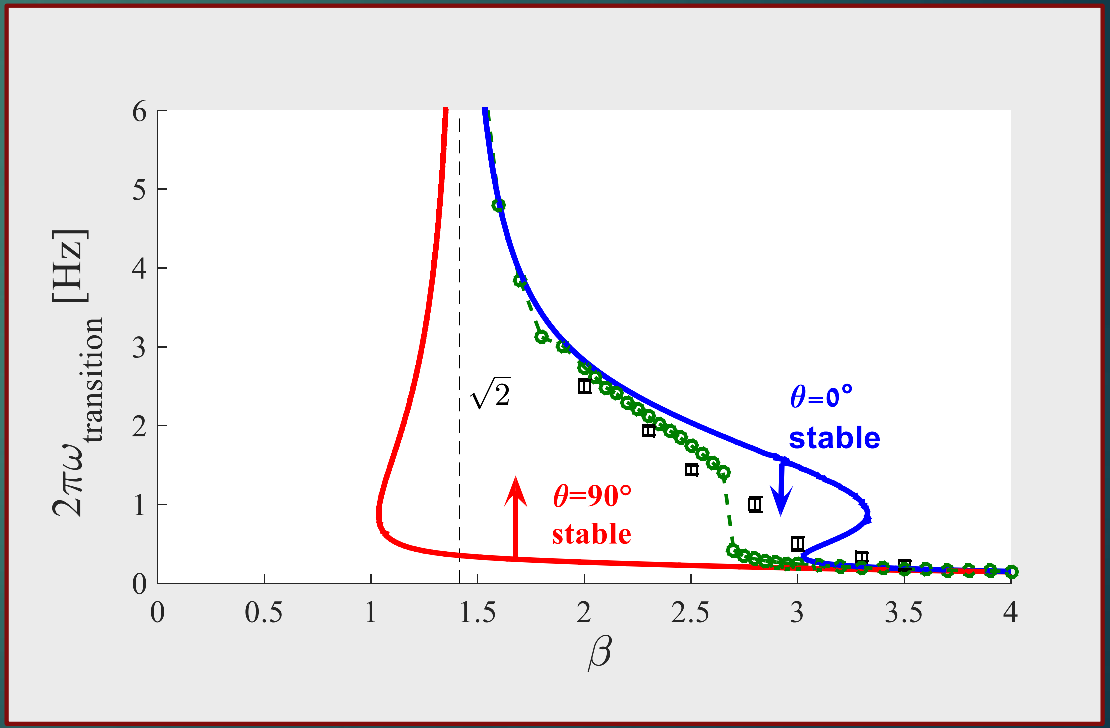
Model fitting

The resultant parameters: $t_m = t_k = 0.1$, no clear asymptotic limit!

Speed vs β



Stability limits



Thank you! Questions?

Contact me: Yuval.Harduf@technion.ac.il