An application of the mean motion problem to time-optimal control

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Motivation

Time-optimal control problems arise in robotics, aerospace, and systems engineering, where reaching a target state quickly is critical.

For **linear systems with scalar input**, the optimal control is known to be *bang-bang*, switching between extreme values.

For systems where all eigenvalues are **real**, the number of switches is bounded by the system dimension.

We study the case where the system has a **purely imaginary spectrum**, and prove that the number of switches must grow linearly with the time horizon.

We relate for the first time the analysis of time optimal control to mean motion problem.

Time Optimal Control

Consider the single-input linear control system

$$\dot{x} = Ax + bu,$$

with $x: [0, \infty) \to \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $u: [0, \infty) \to [-1,1]$.

Optimal Problem: Fix arbitrary $q, p \in \mathbb{R}^n$, and consider the problem of finding a measurable control u, taking values in $[-1,1]$ for all $t \geq 0$, that steers the system from $x(0) = p$ to $x(T) = q$ in **minimal time** T.

Time Optimal Control – Cont.

It is well known that if (A, b) is controllable, such a control u^* exists and satisfies

$$u^*(t) = \mathrm{sgn}\big(m(t)\big)\,,$$

where the switching function $m: [0, T] \to \mathbb{R}$ is given by:

$$m(t) = p^\top(t)b,$$

where the “adjoint state vector” $p: [0, T] \to \mathbb{R}^n \backslash \{0\}$ of the Pontryagin maximum principle satisfies:

$$\dot{p}(t) = -A^\top p(t).$$

Time Optimal Control – Cont.

Typically the value of the adjoint state vector is not known explicitly, so it is natural to study an "abstract" switching function:

$$m(t; p, b, A) = p^\top e^{-At} b.$$

The optimal control satisfies

$$u^*(t) = \begin{cases} 1, & \text{if } m(t) > 0, \\ -1, & \text{if } m(t) < 0. \end{cases}$$

The optimal control is called "bang-bang". The switching points are times t_i such that $m(t_i) = 0$.

Our goal is to determine the number of switching points in the time interval $[0, T]$, denoted $N(T)$.

Main Result

We consider the case where all eigenvalues of A are purely imaginary.

We develop a new approach for analyzing the zeros of $m(t)$ using the classical problem of **mean motion** that was solved by Hermann Weyl in 1938. using this new approach, we showed that on the interval $[0, T]$,

$$N(T) \geq cT \text{ for sufficiently large } T,$$

where c is a positive constant. Our approach also provides a closed-form expression for c in terms of Bessel functions.

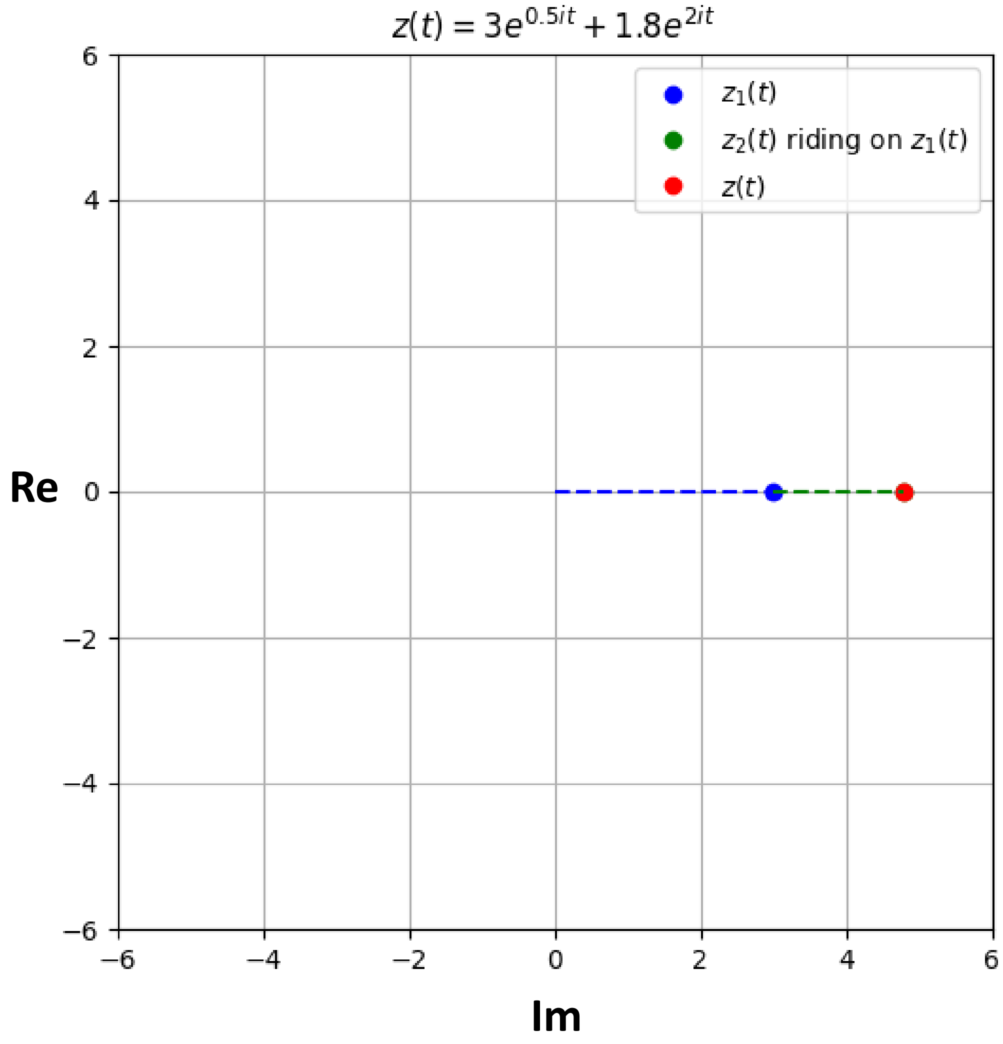
Mean Motion

We first discuss the **mean motion problem** and then relate the problem to time-optimal control.

Consider the complex function $z: [0, \infty) \rightarrow \mathbb{C}$ defined by

$$z(t) := \sum_{k=1}^{n} a_k e^{i(\lambda_k t + \mu_k)},$$

where $a_k \in \mathbb{C}$, $\lambda_k, \mu_k \in \mathbb{R}$. Note that this can be interpreted as a weighted sum of linear oscillators with different frequencies and different phases, or orbits around central bodies.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$z_1(t)$
$z_2(t)$ riding on $z_1(t)$
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Mean Motion – Cont.

Assume that $z(t) \neq 0$ for all $t \geq 0$. Then $\arg\big(z(t)\big) = \arg\big(\sum_{k=1}^{n} a_k e^{i(\lambda_k t + \mu_k)}\big)$ is a continuous function.

The mean motion problem: Determine whether the asymptotic angular velocity of $z(t)$,

$$\Omega \coloneqq \lim_{t\to\infty} \frac{\arg\big(z(t)\big)}{t},$$

exists, and if so, find its value.

This problem goes back to Lagrange who studied the average angular speed of orbiting bodies.

Solution of the Mean Motion Problem

Hermann Weyl proved that

$$\Omega = \sum_{k=1}^{n} \lambda_k V_k,$$

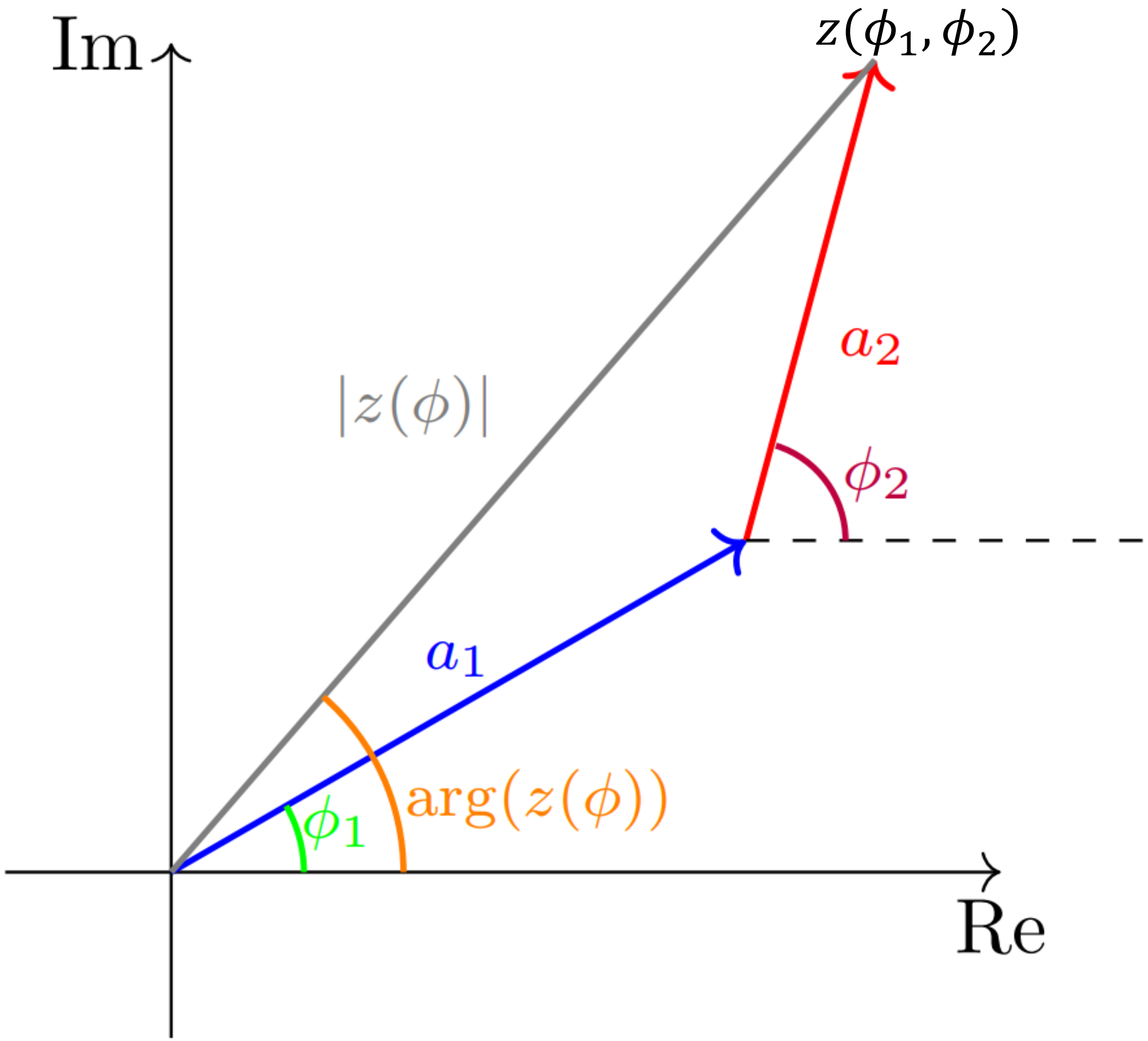
where λ_k is the frequency of oscillator k and $V_k \geq 0$ with $\sum_{k=1}^{n} V_k = 1$.

To explain this result, we first consider the “static” problem.

Let $\phi_k, k = 1, \dots, n$, be angular coordinates. Fix n complex numbers $a_1, \dots, a_n \in \mathbb{C}$. Associate with every set $(\phi_1, \dots, \phi_n)$ a complex number

$$z(\phi_1, \dots, \phi_n) := \sum_{k=1}^{n} a_k e^{i\phi_k}.$$

Geometrically, this can be interpreted as the position of the end-point of a multi-link robot arm where the kth link has length $|a_k|$, and the angle between link k and link $k+1$ is ϕ_k.



Example: For $n = 2$ we have:

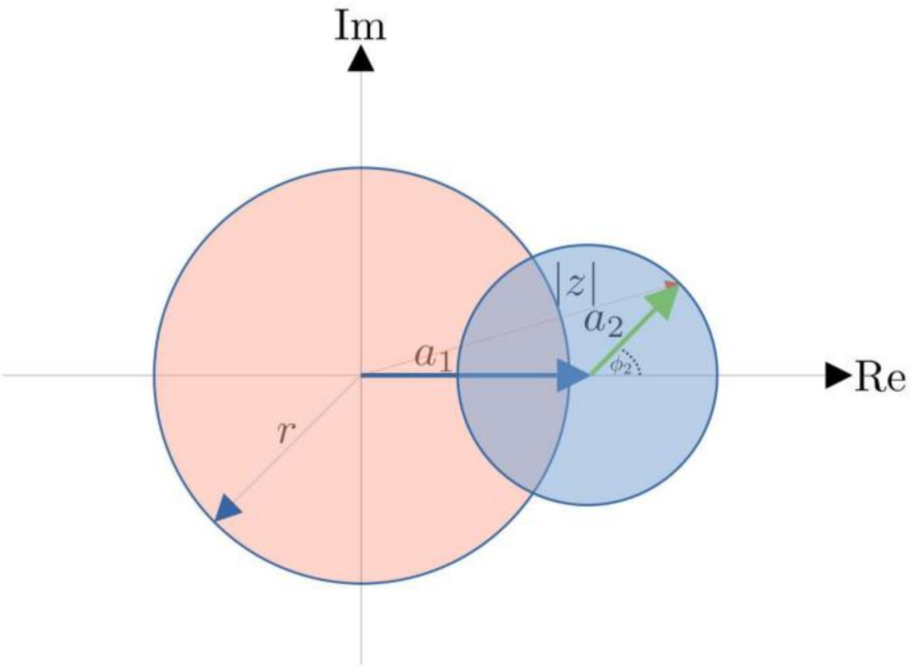
$$z(\phi_1, \phi_2) = a_1 e^{i\phi_1} + a_2 e^{i\phi_2}.$$

Solution of the Mean Motion Problem

For $r \geq 0$. Let $W_n(r) = W_n(r; a_1, \ldots, a_n)$ denote the probability that $|z| = \left|\sum_{k=1}^{n} a_k e^{i\phi_k}\right| \leq r$.
Clearly

$$W_n(r) = \frac{1}{(2\pi)^n} \int_{|z|\leq r} d\phi_1 \ldots d\phi_n \,,$$

the volume of all the angles yielding $|z| \leq r$



Solution of the Mean Motion Problem

Hermann Weyl proved that the mean motion Ω exists, and satisfies

$$\Omega = \sum_{k=1}^{n} \lambda_k V_k,$$

where $V_k \coloneqq W_{n-1}(a_k; a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$. Furthermore, the V_ks are non-negative, and satisfy $\sum_{k=1}^{n} V_k = 1$.

Weyl, Hermann. "Mean motion." *American Journal of Mathematics* 60.4 (1938): 889-896.

Solution of the Mean Motion Problem

Example: Consider the case $n = 2$, that is,

$$z(t) = a_1 e^{i(\lambda_1 t + \mu_1)} + a_2 e^{i(\lambda_2 t + \mu_2)}, \qquad |a_1| > |a_2|.$$

In this case $a_2 e^{i(\lambda_2 t + \mu_2)}$ is a "small perturbation" added to $a_1 e^{i(\lambda_1 t + \mu_1)}$.

The solution of the mean motion problem gives

$$\Omega = \lambda_1 V_1 + \lambda_2 V_2,$$

where V_1 is the probability that $\left|a_2 e^{i\phi}\right| \leq a_1$, that is, $V_1 = 1$. Since $V_1 + V_2 = 1$, $V_2 = 0$.

Thus, in this case

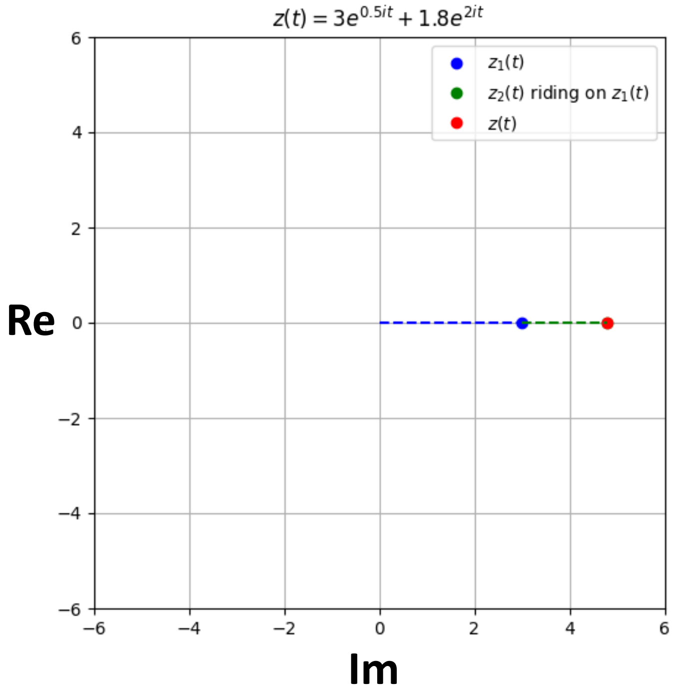
$$\begin{aligned} \Omega &= \lambda_1 V_1 + \lambda_2 V_2 \\ &= \lambda_1 \cdot 1 + \lambda_2 \cdot 0 = \lambda_1. \end{aligned}$$

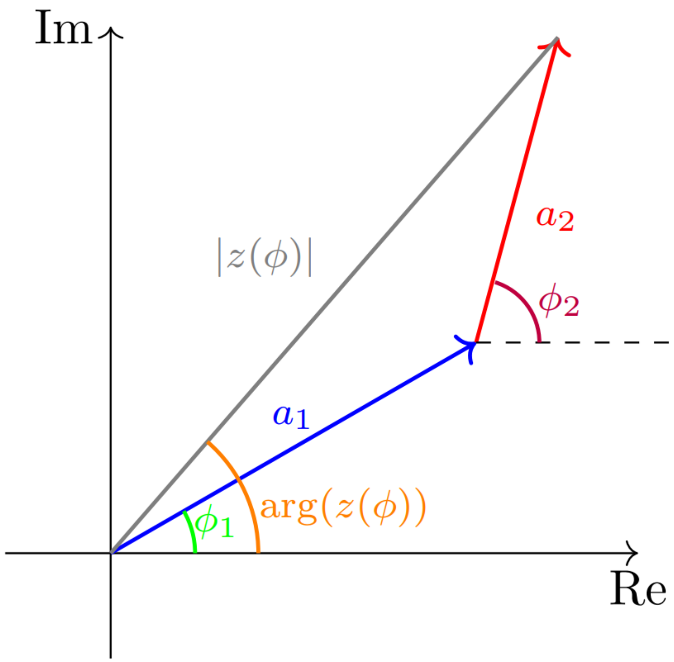
So Far…

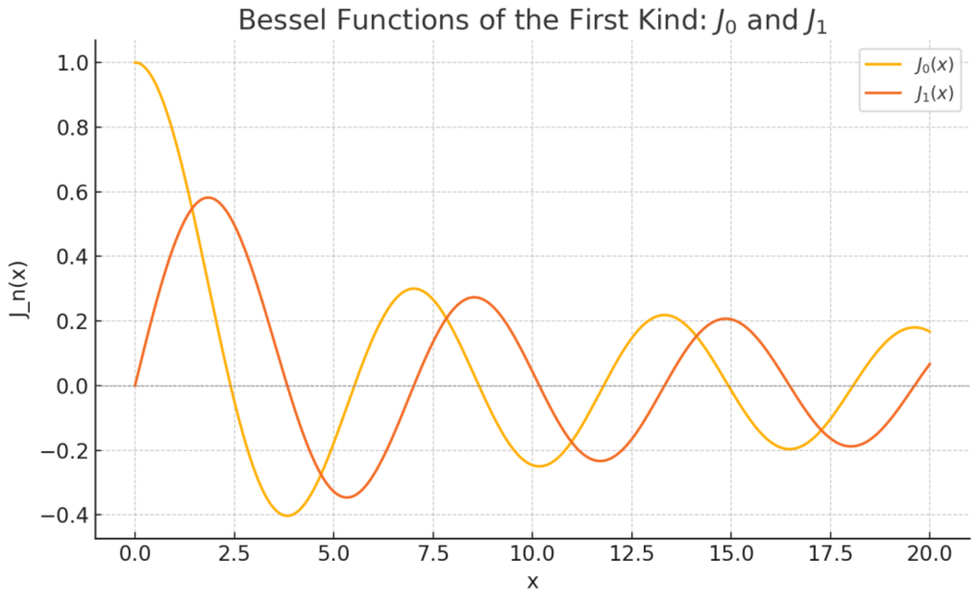
$z(t) \coloneqq \sum_{k=1}^{n} a_k e^{i(\lambda_k t + \mu_k)}$.

$\Omega = \sum_{k=1}^{n} \lambda_k V_k,\, V_k \geq 0.$

Ω as a function of **Bessel** functions.







The Bohl-Weyl-Wintner Formula

The **Bohl-Weyl-Wintner Formula** asserts that

$$W_n(r; a_1, \dots, a_n) = r \int_0^\infty J_1(r\rho) \prod_{k=1}^{n} J_0(|a_k|\rho) d\rho \,,$$

where J_0, J_1 are Bessel functions.

Using the Bohl-Weyl-Wintner formula for W_{n-1} gives

$$\Omega = \sum_{k=1}^{n} \lambda_k a_k \int_0^\infty J_1(a_k\rho) \prod_{k=1}^{n-1} J_0(|a_k|\rho) d\rho \,.$$

Implying that Ω can be computed numerically.

Example of the Mean Motion Problem

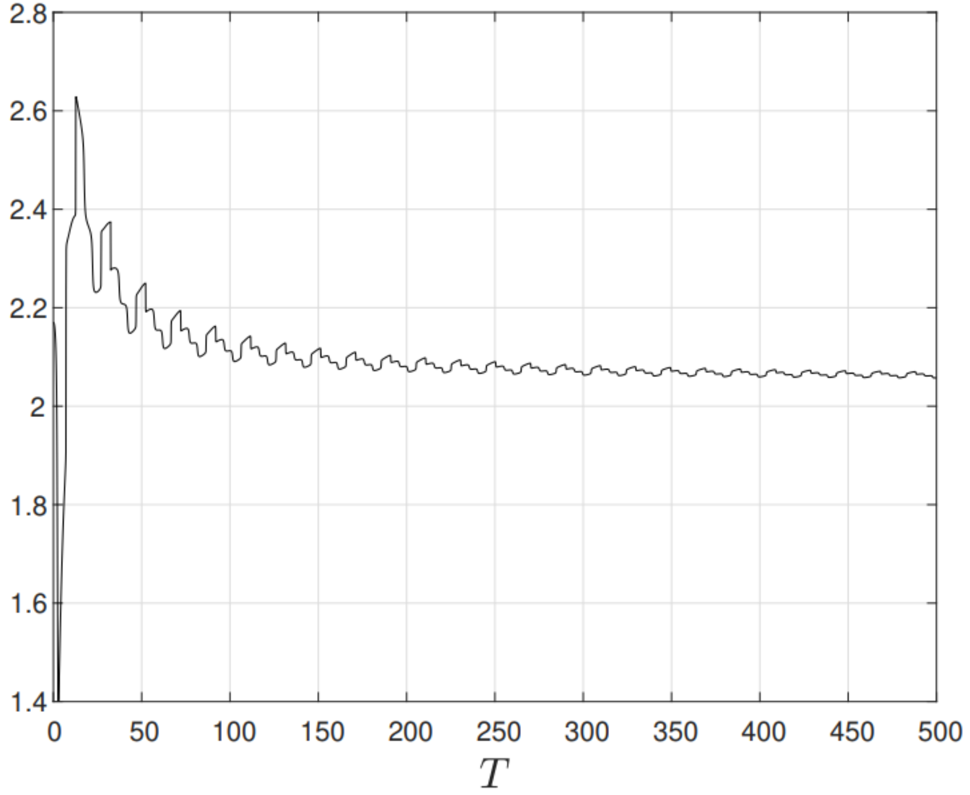
Example: Consider

$$z(t) = e^{i\sqrt{2}t} + 2.5e^{i3t} + 3e^{i\sqrt{3}t}.$$

Then

$$\begin{aligned}
\Omega &= \sum_{i=1}^{3} \lambda_i V_i \\
&= \sqrt{2}W_2(1; 2.5, 3) + 3W_2(2.5; 1, 3) + \sqrt{3}W_2(3; 1, 2.5) \\
&= \sqrt{2}\int_0^\infty J_1(\rho)J_0(2.5\rho)J_0(3\rho)d\rho \\
&+ 7.5\int_0^\infty J_1(2.5\rho)J_0(\rho)J_0(3\rho)d\rho \\
&+ 3\sqrt{3}\int_0^\infty J_1(3\rho)J_0(\rho)J_0(2.5\rho)d\rho.
\end{aligned}$$

A numerical integration gives $\Omega = 2.0614$. Plotting $\frac{\arg(z(t))}{T}$ as a function of T we have



Relating the Number of Switches and the Mean Motion Problem

Optimal Control Problem: steering the system from $x(0) = p$ to $x(T) = q$ in minimal time T.

Solution: We have a switching function: $m(t; p, b, A) = p^\top e^{-At} b$. The optimal Control is

$$u^*(t) = \begin{cases} 1, & \text{if } m(t) > 0, \\ -1, & \text{if } m(t) < 0. \end{cases}$$

Mean Motion Problem: For $z(t) \coloneqq \sum_{k=1}^{n} a_k e^{i(\lambda_k t + \mu_k)}$ we compute $\Omega \coloneqq \lim_{t\to\infty} \frac{\arg(z(t))}{t}$.

Solution: $\Omega = \sum_{k=1}^{n} \lambda_k V_k$, where V_k are given in terms of Bessel Functions.

Our goal is to relate these problems under the assumption that $\sigma(A) \subseteq i\mathbb{R}$.

Relating Switching Function to Mean Motion Problem

The dimension of A is even, and the eigenvalues of $A \in \mathbb{R}^{n \times n}$ are,

$$i\lambda_1, -i\lambda_1, \dots, i\lambda_{\frac{n}{2}}, -i\lambda_{\frac{n}{2}}, \qquad \lambda_i \in \mathbb{R}.$$

We can re-write the switching function:

$$m(t; p, b, A) = p^\top e^{-At} b = \mathrm{Re}\left(\sum_{k=1}^{\frac{n}{2}} a_k e^{i\lambda_k t}\right),$$

where $a_k \in \mathbb{C}$ depend on the entries of b, p and eigenvalues λ_k.

This is the real part of the function that appears in the mean motion problem.

We represent the switching function m as a sum of oscillators.

Example

Consider $A = \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix}, \xi > 0\,, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is easy to confirm that (A, b) is controllable. The eigenvalues of A are $\pm i\xi$. Let $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. Then

$$\begin{aligned} m(t) &= p^\top e^{-At} b \\ &= -p_1 \sin(\xi t) + p_2 \cos(\xi t) \\ &= \mathrm{Re}\Big(-p_1 e^{i\left(\xi t - \frac{\pi}{2}\right)} + p_2 e^{i\xi t} \Big) \\ &= \mathrm{Re}\Big(a_1 e^{i\xi t} \Big), \end{aligned}$$

where $a_1 = -p_1 e^{-\frac{i\pi}{2}} + p_2$.

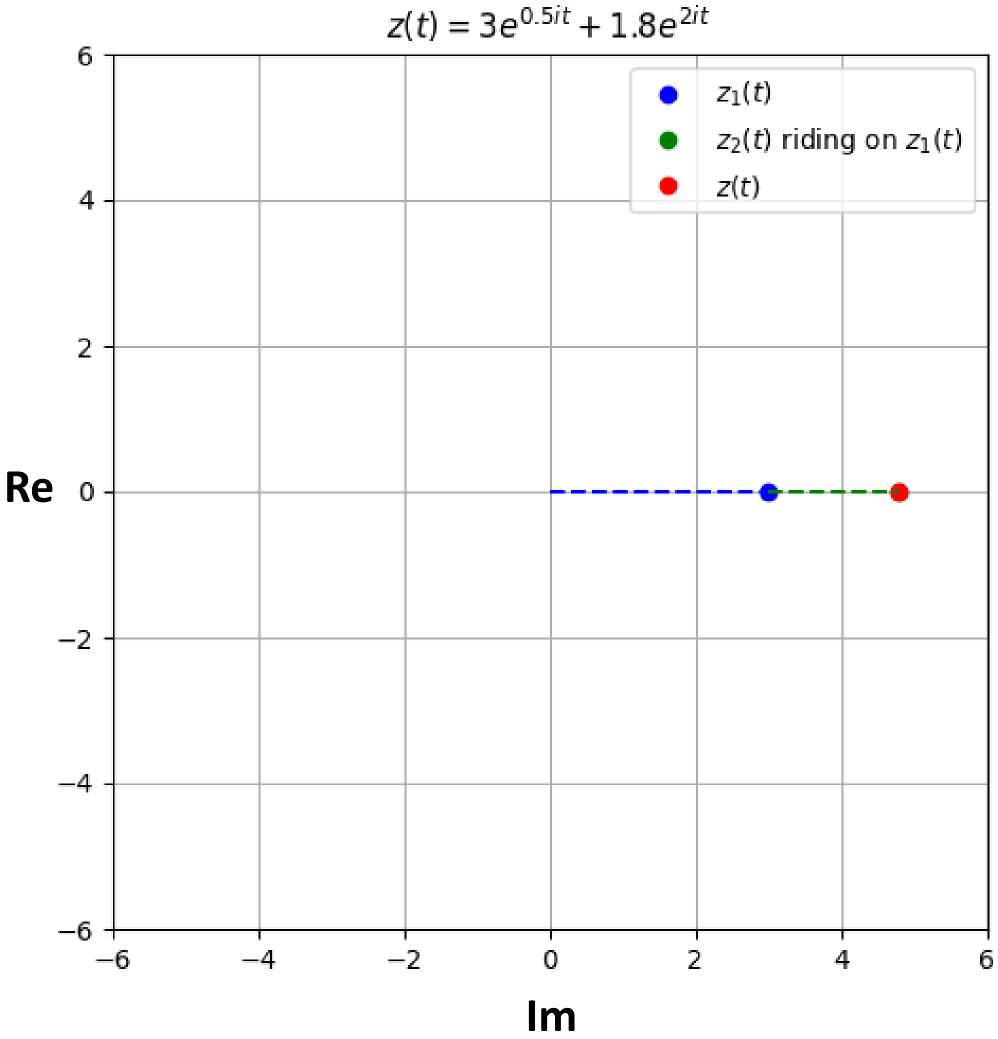
Main Result and Proof

Theorem: Suppose that $\sigma(A) \subseteq i\mathbb{R}$ and that the pair (A, b) is controllable. Then for generic vector $p \in \mathbb{R} \setminus \{0\}$, there exists $c > 0$ such that for any T large enough the number of zeros of the switching function m on the interval $[0, T]$ satisfies

$$N(T) \geq cT + o(T)$$

Proof: We showed that $m(t) = \mathrm{Re}\left(\sum_{k=1}^{\frac{n}{2}} a_k e^{i\lambda_k t}\right)$. Defining $z(t) = \sum_{k=1}^{\frac{n}{2}} a_k e^{i\lambda_k t}$, every time $t_i \geq 0$ such that $\arg\big(z(t_i)\big) = \pi/2$ or $\arg\big(z(t_i)\big) = 3\pi/2$ is a zero of the switching function $m(t)$.

Dalin, Omri, Alexander Ovseevich, and Michael Margaliot. "An application of the mean motion problem to time-optimal control." *arXiv preprint arXiv:2502.13523* (2025).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Proof of Main Result

For T large enough the frequency of $z(t)$ is Ω. This implies that $z(t)$ completes a period every $\frac{|\Omega|}{2\pi}$ units of time. For each period $m(t) = 0$ twice, so the number of zeros of m at time T is:

$$\frac{|\Omega|}{\pi} T + o(T),$$

Since a zero of m corresponds to a switch, we have

$$N(T) \geq \frac{|\Omega|}{\pi} T + o(T).$$

That is, $N(T) \geq cT + o(T)$, where

$$c = \frac{\left| \sum_{k=1}^{n} \lambda_k a_k \int_0^\infty J_1(a_k \rho) \prod_{k=1}^{n-1} J_0(|a_k| \rho) d\rho \right|}{\pi}.$$

Conclusions & Future Research

1) We studies $N(T)$ when $\sigma(A) \subseteq i\mathbb{R}$.

2) We related this problem, for the first time, to the classical mean motion problem.

3) Using this we gave a lower bound on $N(T)$.

<u>Future Research</u>

It is known that if $\sigma(A) \subseteq \mathbb{R}$, then

$$N(T) \leq n - 1 \text{ for all } T > 0.$$

An interesting research direction is to study the case when $\sigma(A) \subseteq \mathbb{C}$.

Thank you

Questions?