Self-sustained oscillations in discrete-time relay feedback systems

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Outline

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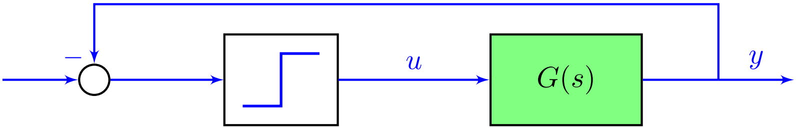
PID-autotuning problem

The PID controller is widely used in engineering, which requires parameter design; *PID autotuning* is an essential method in this context.



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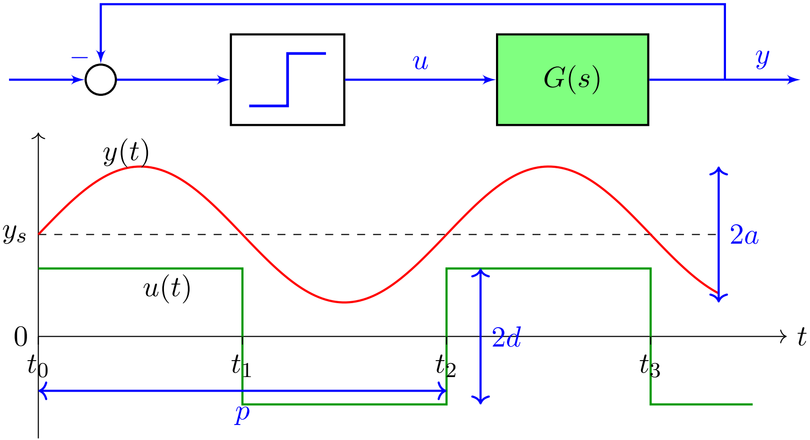

Relay feedback system

For the preparation of matrix analysis,

$$G(s) \implies G(z)$$

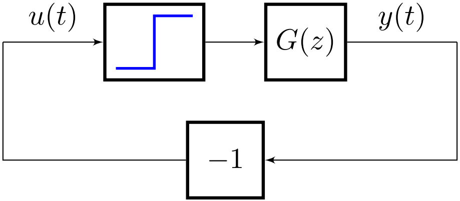

Figure 1: Discrete-time relay feedback system with linear system $G(z)$, $z \in \mathbb{C}$.

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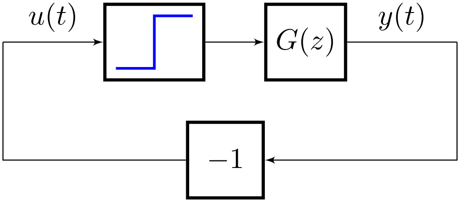

Figure 1: Discrete-time relay feedback system with linear system $G(z)$, $z \in \mathbb{C}$.

Through the discrete *convolution operator* $(\mathcal{C}_g(u)(t) := (g * u)(t))$, this feedback follows

$$u(t) = -\mathcal{C}_g(\text{sign}(u))(t), \quad t \in \mathbb{Z}.$$

Equivalence problem

In the discrete-time domain, the *infinite* operator analysis can be changed into a *finite* operator.

Convolution operator form

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$$u^P = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(P-1) \end{bmatrix}, \; H_{\overline{g}^P} := \begin{bmatrix} \overline{g}_1^P & \overline{g}_P^P & \cdots & \overline{g}_2^P \\ \overline{g}_2^P & \overline{g}_1^P & \cdots & \overline{g}_3^P \\ \vdots & \vdots & & \vdots \\ \overline{g}_P^P & \overline{g}_{P-1}^P & \cdots & \overline{g}_1^P \end{bmatrix} \in \mathbb{R}^{P\times P}, \; \overline{g}_i^P := \sum_{j=-\infty}^{\infty} g(i-1+jP).$$

Assumptions and our goals

In this relay feedback system, we study the *sustained unimodal oscillations* in discrete-time relay feedback systems having the following input-output property.

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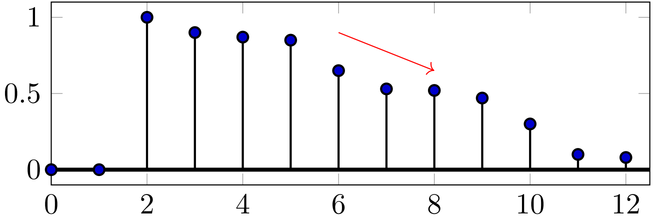

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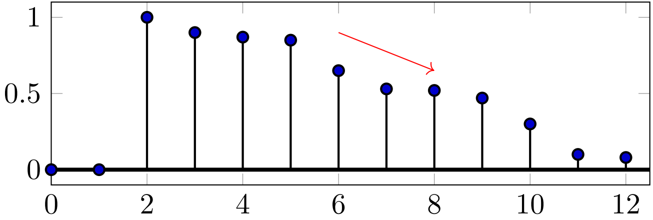


We focus on the following questions:

- Necessary conditions for self-sustained oscillations
- Absence of self-oscillations
- Period of self-oscillations

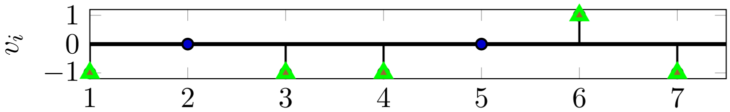
Outline

Unimodality and variation

Goal: Describing unimodality in periodic sequence in math.

- **Variation of a vector:** For a vector $v \in \mathbb{R}^n$,

$$S^-(v) := \#\text{ of sign changes (after deleting zeros)}$$



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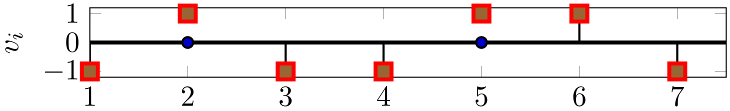

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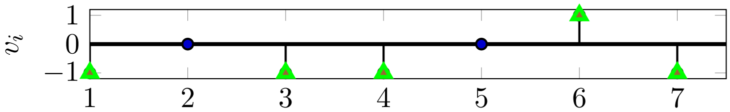

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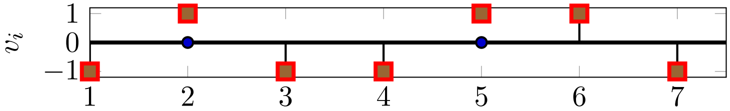

- **Cyclic variation of a vector:** For a vector $v \in \mathbb{R}^n$, $*$ represents both $-$ and $+$,

$$S_c^*(v) := \begin{cases} S^*(v) & \text{when } S^*(v) \text{ even,} \\ S^*(v)+1 & \text{when } S^*(v) \text{ odd.} \end{cases}$$

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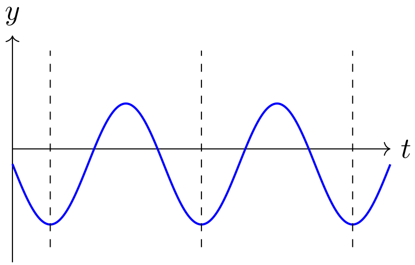
Observation: The fundamental period of a unimodal oscillation.



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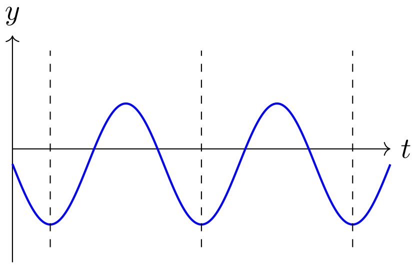

choose one period
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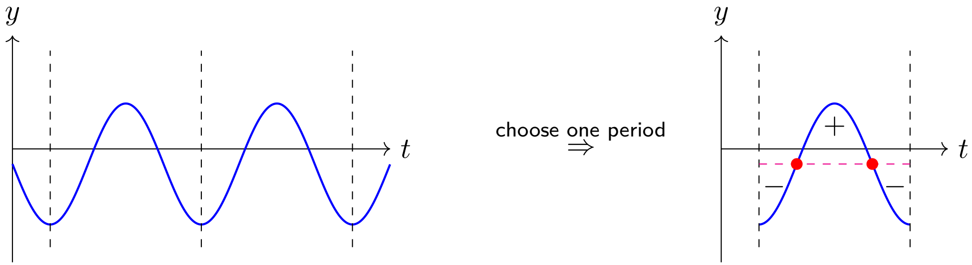

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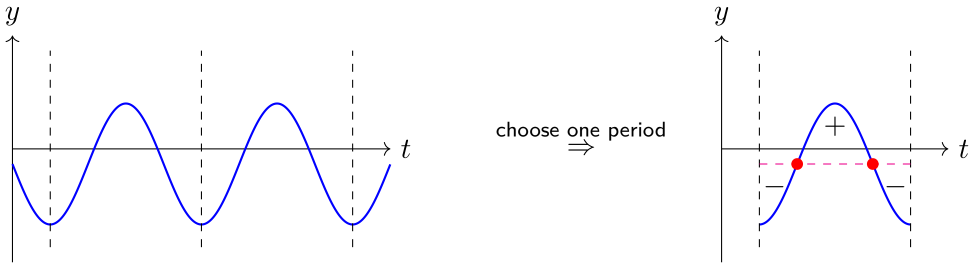


Note that the "variation" of its fundamental period under any vertical shift is at most two.

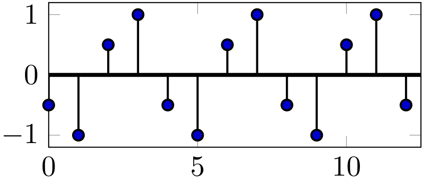
Invariance and variation

Goal: Find something invariant in a periodic oscillation.

Observation: Cyclic variation of any one period is invariant.

- Invariance and unimodality
  - u is unimodal, i.e.,

$$S_c^-(\Delta_c u^P) = 2 \text{ and } S_c^+(u^P) = 2$$



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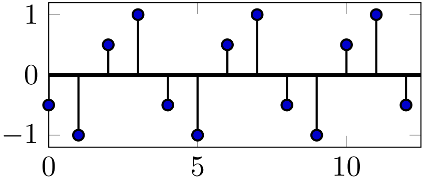

$*$

$\Downarrow$

  - g is strictly monotonic decreasing, i.e.,

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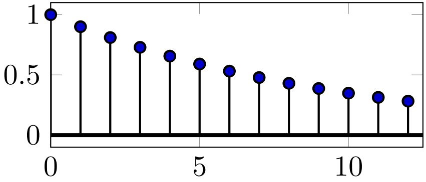

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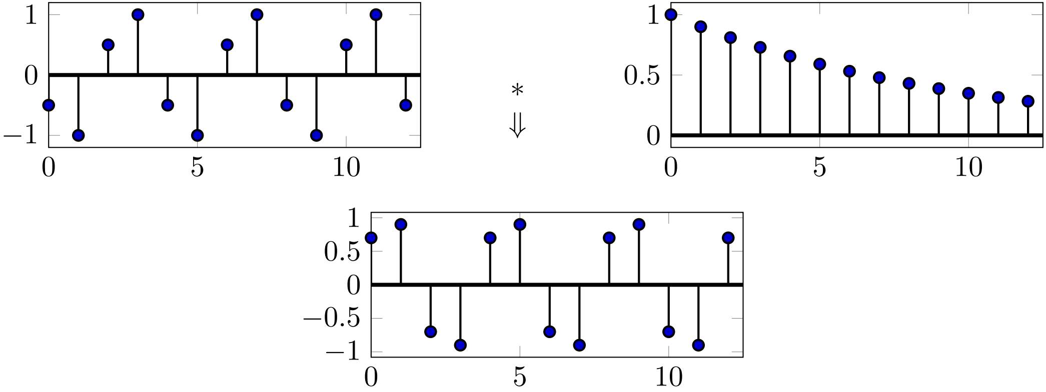

Outline

Main Results

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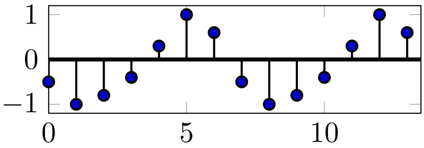

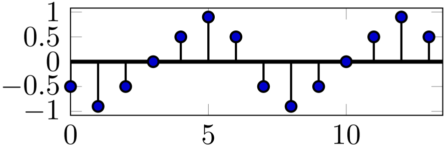

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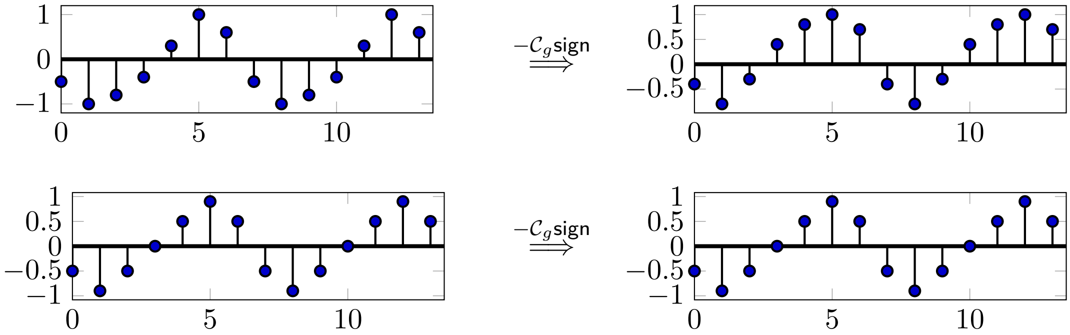

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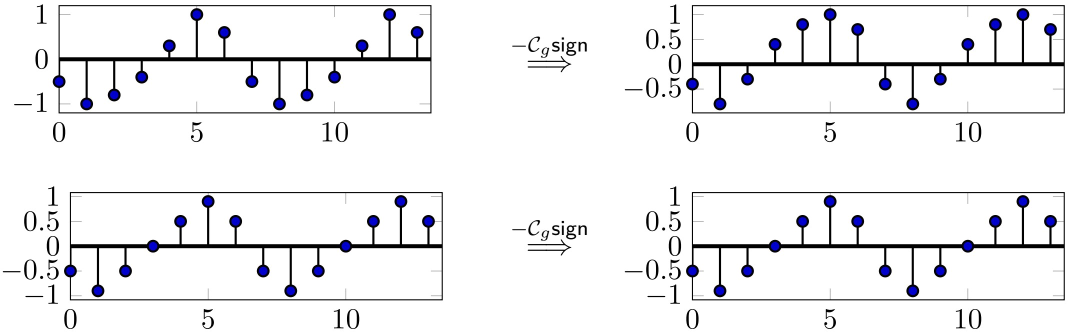
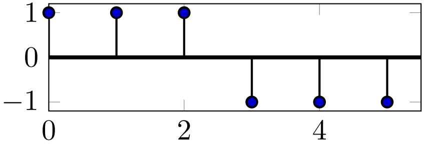


$\Longrightarrow$ the number of positive elements $=$ the number of negative elements

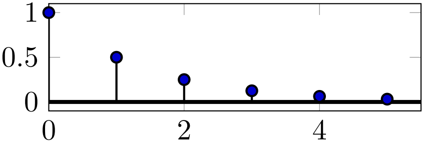
Main Results

2. Absence of self-oscillations

- $\text{sign}(u^6) = [1, 1, 1, -1, -1, -1]^\top$



- $\overline{g}^6 = [1, 0.5, 0.25, 0.12, 0.06, 0.03]^\top$



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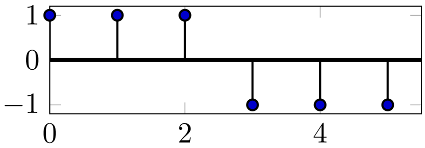

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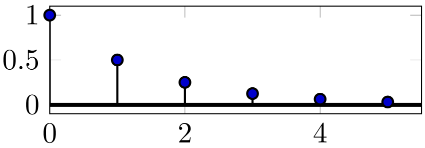

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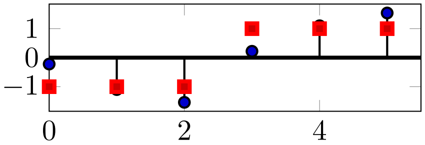
$$u^6 = -H_{\overline{g}^P}\mathsf{sign}(u^P)$$
$$= [-0.22, -1.1, -1.54, 0.22, 1.1, 1.54]^\top$$



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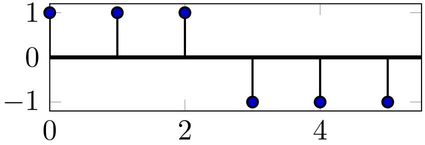

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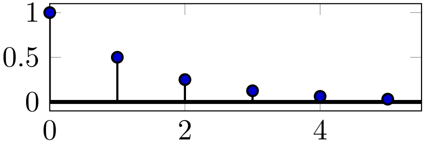

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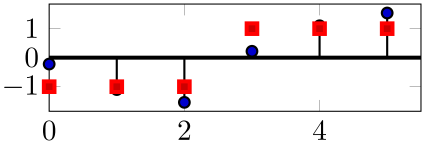

$$g(0) > 0 \Longrightarrow u^P \neq -H_{\bar{g}^P} \text{sign}(u^P) \text{ for any } P.$$

Main Results

3. Self-oscillations under time delays

$$G(z) = z^{-P_d} G_0(z) \text{ where } P_d \geq 0$$

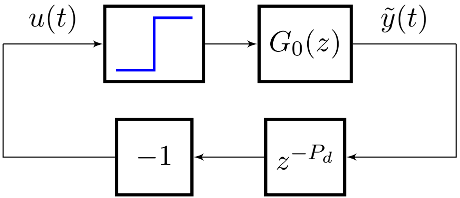

Figure 2: A DT Relay Feedback System with delay module.

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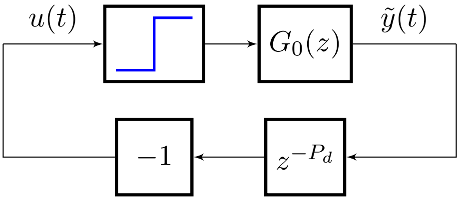

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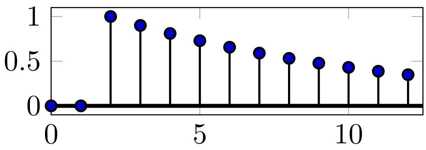

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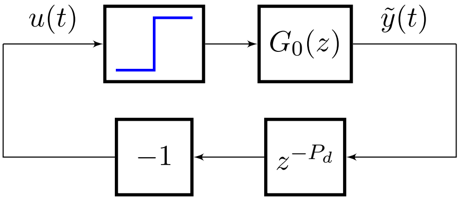

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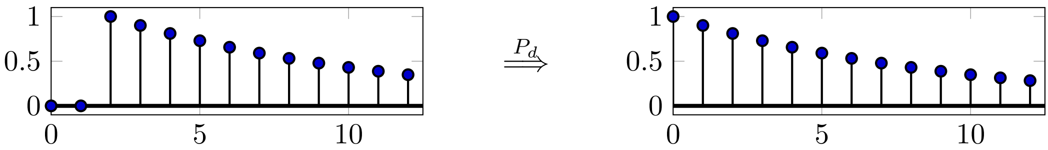
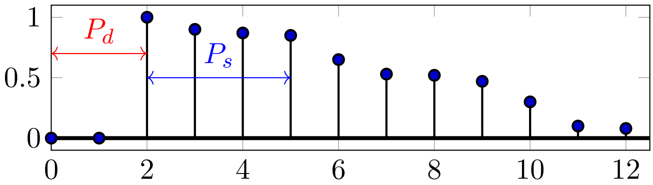


Main Results

3. Self-oscillations under time delays: the bound for the self-oscillation

- P_d - delay time
- P_s - main part of response, i.e., $\arg\min_{P_s} \sum_{i=P_d}^{P_d+P_s-1} g(i) > \sum_{i=P_d+P_s}^{\infty} g(i)$



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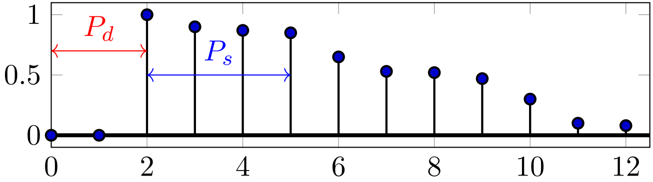
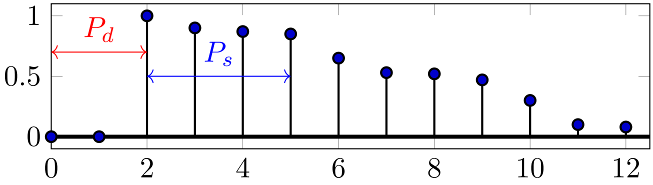


$$\Rightarrow \quad 2P_d \leq P \leq 2(P_d + P_s).$$

Corollary

3. Self-oscillations under time delays: other self-oscillations



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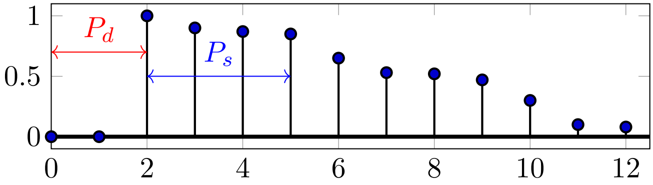

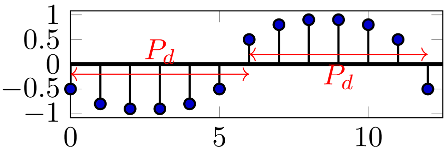

If a self-oscillation with $P = 2P_d$ is admitted.

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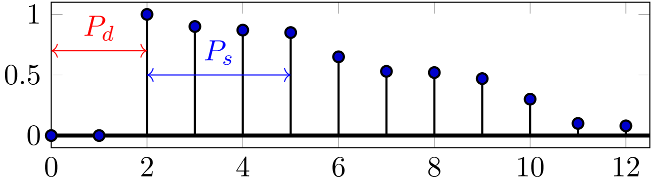
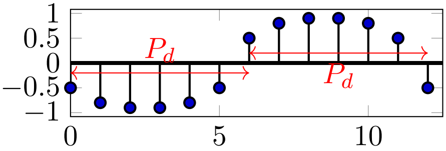
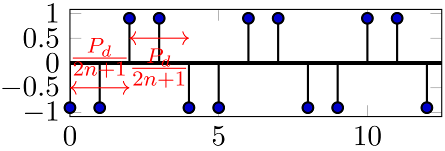




$\Longrightarrow$



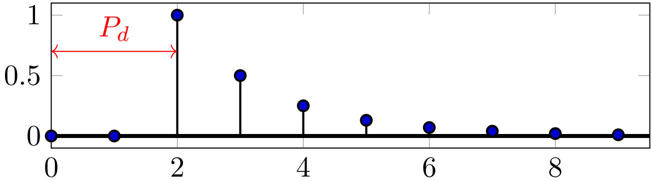
If a self-oscillation with $P = 2P_d$ is admitted.

Then a self-oscillation with $P = \frac{2P_d}{2n+1}$ is also admitted.

Corollary

3. Self-oscillations under time delays: the convex impulse response

- P_d - delay time
- $g(t)$ - convex function



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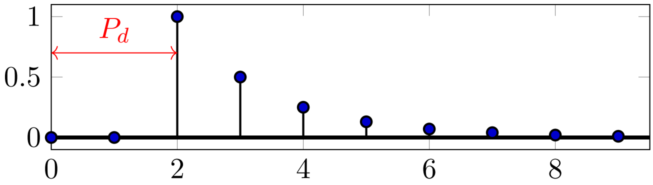

$$\Rightarrow \quad 2P_d \leq P \leq 4P_d + 2.$$

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In the future, it would also be interesting to explore

- Higher variation oscillations.
- Non-monotone impulse responses.