An Explicit Solution to the LQ Tracking Problem with Applications to Optimal Controllers Based on Machine Learning

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Conceptual shift in the approaches for power systems control and management

- The centralized approach is slowly being replaced by the decentralization of knowledge and control span among different entities
- Complex control problems and introducing a large number of decision variables
- Moreover, the resulting optimization problems are often not convex
- Emerging trend of using machine-learning methods for power systems applications

Can we make it better?

- Using knowledge about systems dynamics to reduce computational complexity required for solving these problems.

Tracking problems

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax(t) + Bu(t)$$

$$J = \frac{1}{2T}\int_0^T \big(x(t) - x_m(t)\big)^{\mathrm{T}} Q\big(x(t) - x_m(t)\big) + \big(u(t) - u_m(t)\big)^{\mathrm{T}} R(u(t) - u_m(t))\mathrm{d}t$$

- Initial state $x(0) = x_0$,
- The objective is to find the optimal control law $u^*(t)$ that minimizes the cost J.

- Resembles LQR formulation, but depends on future values of $x_m(t), u_m(t)$

Tracking problems are well studied, so what's new?

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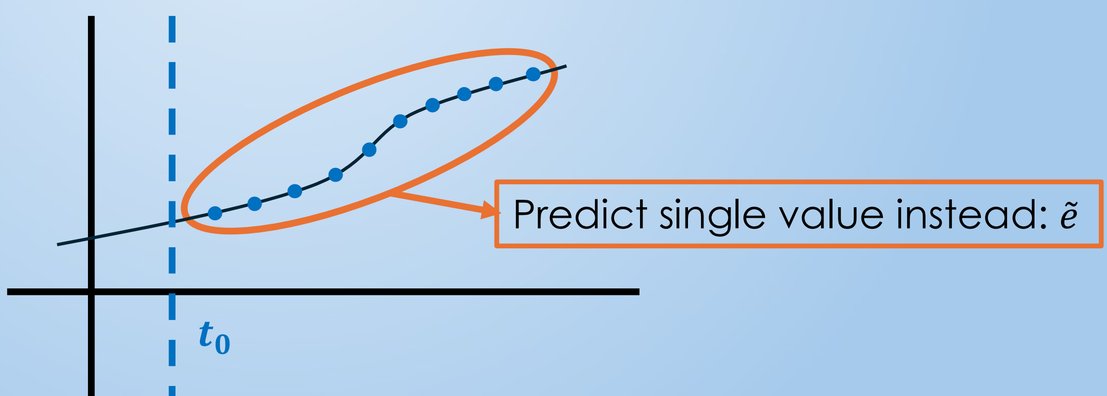
- Known reference signals (dynamic programming, quadratic programming)
- Unknown reference signals – simple statistics (stochastic dynamic programming)
- Unknown reference signals – complicated statistics: We want to predict the reference signal
- Possible solution: Using a machine learning estimator

Predict the reference signal

- We need to train the model on large number of long time-series, resulting in high computational complexity
- Do we have to predict the whole remaining part of the trajectory at every moment in time? **No!**
- The proposed solution is based on an explicit integral of the reference signals, thus it reduces considerably the dimension of the solution space
- We can now estimate a single value representing the future trajectory at any given moment



Main Result:

- To obtain the optimal control law, for infinite time horizon and free final point, we apply Pontryagin's Minimum Principle
- This results in a set of N linear equations and N free variables (in λ_0^*). These may be solved to obtain λ_0^*. The solution is characterized by the Theorem:

Theorem 1: Assume that $Q \geq 0$, $R > 0$, the pair (A, B) is stabilizable, the pair (A, Q) is detectable, H is diagonalizable, V_1, V_4 are invertible, and the signals $u_m(t), x_m(t)$ are bounded and piecewise continuous. Then

1)

$$Pr\,(T) - q\,(T) = \int_0^T \exp(A_0^\mathsf{T}\tau)\,[P \quad -I]\, G(\tau)\mathrm{d}\tau.$$

2) There exists a constant $T_{\min}$ such that for every $T \geq T_{\min}$, (11) has a unique solution $\lambda_0{}^*$.
3) This unique solution $\lambda_0{}^*$ has the following property:

$$\lim_{T\to\infty} \lambda_0{}^* = Px_0 + \int_0^\infty \exp(A_0^\mathsf{T}\tau)\,[P \quad -I]\, G(\tau)\mathrm{d}\tau.$$

Main Result:

- Focusing on bullets number (2) and (3):
- We prove that the obtained solution λ_0^* is unique
- Following, in bullet number (3) we present the explicit expression of the solution λ_0^* at the limit when T tends to infinity

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The optimal control law

Now, we can use this unique solution λ_0^* to obtain an explicit expression of the optimal control law for the tracking problem, at the limit $T \to \infty$

$$\lim_{T\to\infty} u^*(t) = u_m(t) - R^{-1}B^{\mathsf{T}}Px^*(t) - R^{-1}B^{\mathsf{T}} \cdot \int_0^{\infty} \exp\left(A_0{}^{\mathsf{T}}\tau\right)\left(PBu_m(t+\tau) - Qx_m(t+\tau)\right)d\tau.$$

Example: Energy storage device management

$$\frac{\mathrm{d}}{\mathrm{dt}}E = p_g(t) - p_l(t) \text{ with } E(0) = x_0,$$

$$J = \frac{1}{2}\int_0^T \alpha^2 (E(t) - \frac{1}{2}E_{\max})^2 + p_g^2(t)\mathrm{d}t.$$

$$x(t) = E(t),$$

$$u(t) = p_g(t) - p_l(t),$$

$$u_{\mathrm{m}}(t) = p_l(t).$$

What should we expect? Shortest Path method

- Graphical procedure that results in the optimal solution for the case of an ideal energy storage device
- Plot the energy bands $E_l, E_l + E_{\max}$
- The optimal generated energy is the shortest path between the points $E_l(0)$ and $E_l(T)$
- The generated power should approximate the average load power consumption due to the energy storage device contribution

Y. Levron and D. Shmilovitz, "Optimal Power Management in Fueled Systems With Finite Storage Capacity," in IEEE Transactions on Circuits and Systems I: Regular Papers, vol. 57, no. 8, pp. 2221-2231, Aug. 2010, doi: 10.1109/TCSI.2009.2037405.

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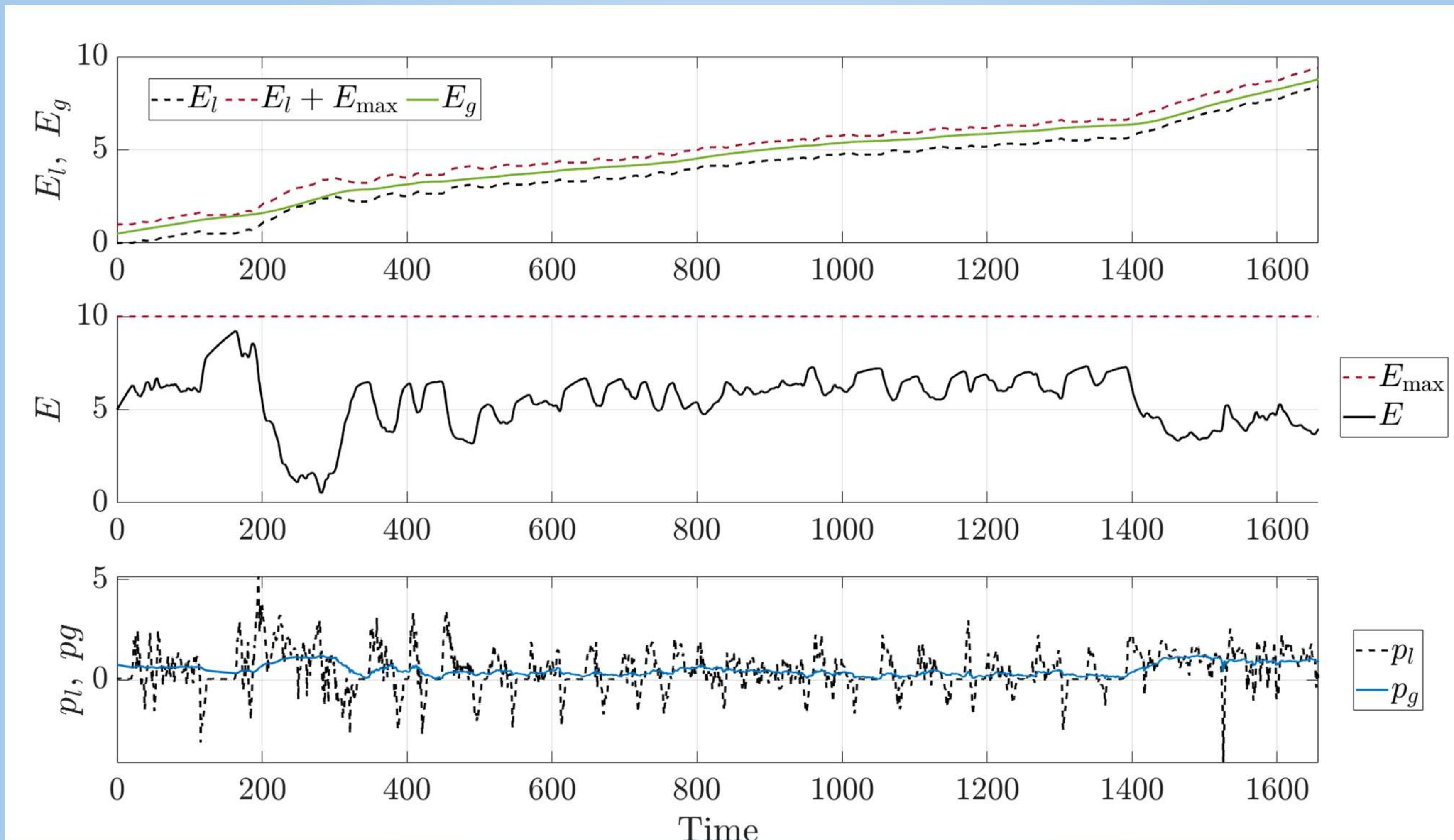

Conclusions

- Solving Linear Quadratic tracking problems is challenging when the reference signals are unknown: The optimal control depends on the entire future reference trajectory
- Machine learning offers a potential solution for tracking unknown reference signals by predicting future trajectories, but it increases computational complexity
- This work present a solution to the LQ tracking problem in a new form, that relies on an explicit integral of the reference signals
- When the reference signals are **known**, the explicit solution may be advantageous for problems with large number of decision variables
- When the reference signals are **unknown**, and have complicated underlying statistical model, this solution leads to an efficient training process of machine learning estimators
- Optimal tracking problems remain an important and exciting area of research for both researchers and engineers, and this work contributes to the ongoing research