Quantitative Stability of Autonomous Linear Systems

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Undergraduate Background

Continuous-time | **Discrete-time**

$$x \in \mathbb{R}^n$$

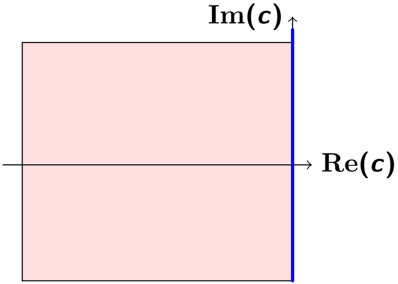
$$t \geq 0 \qquad\qquad k = 0,\ 1.\ 2,\ \ldots$$

$$\dot{x}(t) = A_c x(t) \qquad\qquad k(k+1) = A_d x(k)$$

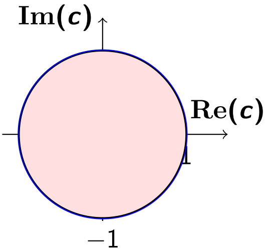
$$x(t) = e^{A_c t} x(0) \qquad\qquad x_d(k) = {A_d}^k x(0)$$

exponential stability

$$\text{spec}(A_c) \subset \mathbb{C}_L \qquad\qquad \text{spec}(A_d) \subset \mathbb{D}(0,\ 1)$$



$\mathbb{C}_L$



$\mathbb{D}(0,\ 1)$

end of undergraduate background

$\exists\ P \succ 0$

The Stein Inclusion

$$\mathbf{S}_P := \{A_d \in \mathbb{C}^{n\times n} \ : \ P - {A_d}^* P A_d \succ 0\}$$

$$A_d \in \mathbf{S}_P \iff \mathrm{spec}(A_d) \subset \mathbb{D}(0,\ 1)$$

The Lyapunov Inclusion

$$\mathbf{L}_P := \{A_c \in \mathbb{C}^{n\times n} \ : \ -(PA_c + {A_c}^* P) \succ 0\}$$

$$A_c \in \mathbf{L}_P \iff \mathrm{spec}(A_c) \subset \mathbb{C}_L$$

The Cayley Transform

$A_d, A_c \in \mathbb{C}^{n\times n} \quad -1 \notin \mathrm{spec}(A_d) \quad 1 \notin \mathrm{spec}(A_c)$

$$A_d = \mathcal{C}_c(A_c) := (I_n + A_c)(I_n - A_c)^{-1}$$

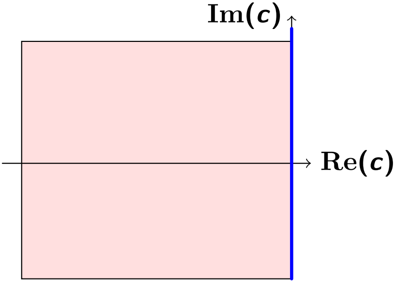
$$A_c = \mathcal{C}_d(A_d) := (A_d - I_n)(A_d + I_n)^{-1}$$

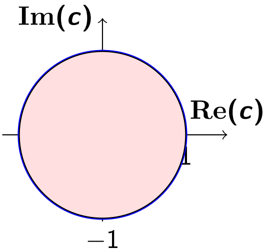
properties (whenever inverses exist)

$$\mathcal{C}_c(\mathcal{C}_d(A_d)) = A_d \qquad \mathcal{C}_d(\mathcal{C}_c(A_c)) = A_c$$

$$\mathcal{C}_c(A_c^{-1}) = -A_d \qquad \mathcal{C}_d(-A_d) = A_c^{-1}$$

$$\mathcal{C}_c(\underbrace{\mathbb{C}_L}_{\text{left half plane}}) = \underbrace{\mathbb{D}(0,\ 1)}_{\text{unit disk}} \qquad \mathcal{C}_d(\underbrace{\mathbb{D}(0,\ 1)}_{\text{unit disk}}) = \underbrace{\mathbb{C}_L}_{\text{left half plane}}$$





The Stein and the Lyapunov Inclusions (again)

$\exists \ P \succ 0$

The Stein Inclusion

$$\mathbf{S}_P := \quad \{A_d \in \mathbb{C}^{n \times n} \ : \ P - {A_d}^* P A_d \succ 0\}$$

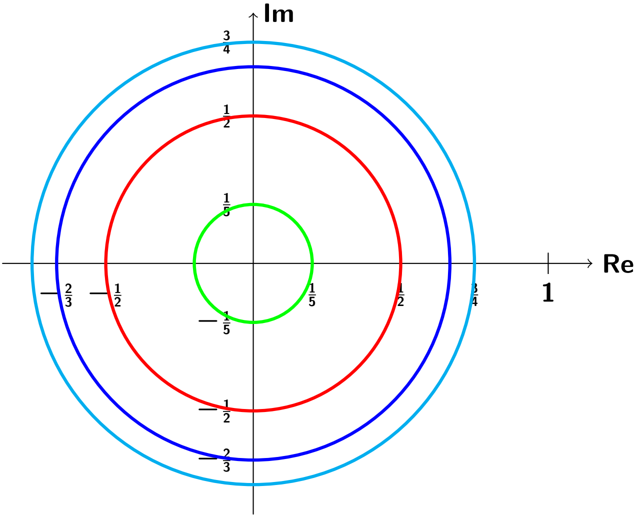
The Lyapunov Inclusion

$$\mathbf{L}_P := \quad \{A_c \in \mathbb{C}^{n \times n} \ : \ -(P A_c + {A_c}^* P) \succ 0\}$$

The Cayley transform

$$\mathcal{C}_d(\mathbf{S}_P) = \mathbf{L}_P \qquad\qquad \mathcal{C}_c(\mathbf{L}_P) = \mathbf{S}_P$$

Quantitative Stability - Sub-Unit Disk



$$\mathbb{D}\left(0,\ \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right) \quad : \quad \beta = \frac{7}{25} \quad \beta = \frac{5}{13} \quad \beta = \frac{3}{5} \quad \beta = \frac{12}{13}$$

Quantitative Hyper-Stein Inclusion

$\exists\ P \succ 0 \qquad\qquad P - A_d{}^* P A_d \succ 0$

For $\beta \in [0,\ 1) \qquad\qquad (1-\beta)P - A_d{}^*(1+\beta)PA_d \succ 0$

$$\mathbf{S}_P(\beta) := \{A_d \in \mathbb{C}^{n\times n}\ :\ P - A_d{}^* P A_d \succ \beta(P + A_d{}^* P A_d)\}$$

$$A_d \in \mathbf{S}_P(\beta) \iff \operatorname{spec}(A_d) \subset \mathbb{D}\left(0, \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right)$$

$1 > \beta > \hat{\beta} > 0 \implies \mathbf{S}_P(\beta) \subset \mathbf{S}_P(\hat{\beta}) \qquad\qquad \lim_{\beta \longrightarrow 0} \mathbf{S}_P(\beta) = \mathbf{S}_P$

[L. (2024)]

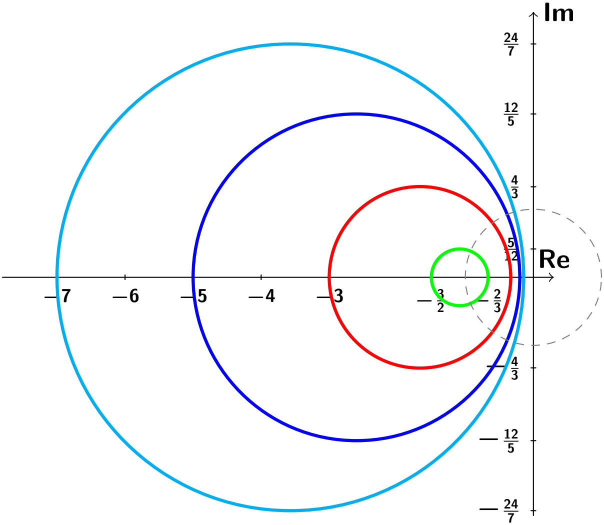
$$\mathcal{C}_d\left(\mathbb{D}\left(0,\ \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right)\right) = \underbrace{\mathbb{D}\left(\frac{1}{\beta}+i0,\ \frac{\sqrt{1-\beta^2}}{\beta}\right)}_{\mathbb{D}_{\mathrm{INV}}(\beta)} \qquad \beta \in [0,\ 1)$$

$$\mathcal{C}_d\left(\mathbb{D}\left(0,\ 1\right)\right) = \mathbb{C}_L \qquad \beta = 0$$

$$\left(\mathbb{D}\left(\frac{1}{\beta}+i0,\ \frac{\sqrt{1-\beta^2}}{\beta}\right)\right)^{-1} = \left(\mathbb{D}_{\mathrm{INV}}\left(\beta\right)\right)^{-1} = \mathbb{D}_{\mathrm{INV}}\left(\beta\right)$$

A. Rantzer, 1993 : A Weak Kharitonov Theorem holds iff the Stability Region and its Reciprocal are Convex

The Cayley Transform of a Sub-Unit Disk (cont.)



$\mathbb{D}_{\mathrm{INV}}(\beta)$ $\beta = \frac{7}{25}$ $\beta = \frac{5}{13}$ $\beta = \frac{3}{5}$ $\beta = \frac{12}{13}$

Quantitative Hyper-Lyapunov Inclusion

$\exists P \succ 0 \quad \beta \in [0,\ 1)$

$$\mathcal{C}_d(\mathbf{S}_P(\beta)) = \mathbf{L}_P(\beta) = \{A_c \in \mathbb{C}^{n \times n} : PA_c + A_c{}^* P \succ \beta(P + A_c{}^* P A_c)\}$$

$$A_c \in \mathbf{L}_H(\beta) \quad \Longleftrightarrow \quad \mathbf{spec}(A_c) \subset \mathbb{D}_{\mathrm{INV}}(\beta)$$

$$1 > \beta > \hat{\beta} > 0 \quad \Longrightarrow \quad \mathbf{L}_H(\beta) \subset \mathbf{L}_H(\hat{\beta}) \qquad \lim_{\beta \longrightarrow 0} \mathbf{L}_H(\beta) = \mathbf{L}_H$$

An Application - Differential Inclusion

The Hyper-Lyapunov Inclusion $\quad P \succ 0 \quad \beta \in [0,\ 1)$

$$\mathbf{L}_P(\beta) := \{A_c \in \mathbb{C}^{n\times n} \ : \ PA_c + {A_c}^*P \succ \beta(P + {A_c}^*PA_c) \ \}$$

A Differential Inclusion: For a given finite set $\mathrm{M} \subset \mathbb{C}^{n\times n}$

$$\dot{x} \in \mathrm{M}x \quad \text{means}$$

$\dot{x} = A(\cdot,\cdot)x$ dependence on t, x arbitrary, but $A(\cdot,\cdot) \in \mathrm{M}$

$\exists \ P \succ 0$ s.t. $\mathrm{M} \subset \mathbf{L}_P(\beta) \implies$

all trajectories are quantitatively bounded from above and below

References

Main:

I.L. "On Hyper-Lyapunov Matrix Inclusions", Linear Algebra and its Applications, Vol. 694, pp. 414-440, 2024

Background

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I.L. " Passive Linear Systems Discrete-time: Characterization through Structure", Linear Algebra and its Applications, No. 15718, 2021.

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Applications:

D. Alpay, I.L. "Quantitatively Hyper-Positive Real Functions", Linear Algebra and its Applications, Vol. 623, pp. 316-334, 2021

D. Alpay, I.L. "Quantitatively Hyper-Positive Real Functions II", Linear Algebra and its Applications, Vol. 697 pp. 332-364, 2024

D. Alpay, I.L. "Quantitatively Hyper-Positive Real Functions III", submitted

THANKS FOR YOUR ATTENTION