On the Gain of Entrainment in Contractive Control Systems

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Entrainment and Gain of Entrainment

A system entrains if for any T-periodic input its output converges to a unique T-periodic solution.

Examples: biological organisms entrain to the 24 hours solar day

synchronous generators entrain to the frequency of the electric grid

A system admits a gain of entrainment (GOE) if applying a periodic control leads to a “better” output, on average, than the equivalent constant control.

This property is important when we can control a system using either a constant control or a periodic control, and we would like to maximize the average production rate.

Entrainment in Stable Linear Systems

Consider a SISO stable linear control system:

$$u \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= c^T x(t) \end{aligned}} \longrightarrow y$$

Let $G(s) = c^T(sI - A)^{-1}b$. Compare two inputs:

$$u(t) = v \longrightarrow \boxed{G} \longrightarrow y_{ss}(t) = G(0)v$$

$$u(t) = v + \mathrm{a} * \sin(\omega t) \longrightarrow \boxed{G} \longrightarrow y_{ss}(t) = G(0)v + a|G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Linear Systems Have no GOE

$$u(t) = v \rightarrow \boxed{G} \longrightarrow y_{ss}(t) = G(0)v$$

$$u(t) = v + \mathrm{a} * \sin(\omega t) \rightarrow \boxed{G} \longrightarrow y_{ss}(t) = G(0)v + a|G(i\omega)|\sin(\omega t + \angle G(i\omega))$$

The two controls have equal average

The two steady-state outputs have equal average

Conclusion: no gain in using a periodic control over a constant one, so no GOE.

GOE in a Nonlinear System

Consider the nonlinear system:

$$u \longrightarrow \boxed{\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u \\ y &= x_1 \end{aligned}} \longrightarrow y$$

This system entrains. Consider two inputs:

$$u(t) = 0 \rightarrow \boxed{\text{NS}} \longrightarrow y_{ss}(t) = 0$$

$$u(t) = a\sin(\omega t) \rightarrow \boxed{\text{NS}} \longrightarrow y_{ss}(t) = \cdots, \qquad \frac{1}{T}\int_0^T y_{ss}(t)dt = \frac{a^2}{2(1+\omega^2)}.$$

The system has a GOE that depends on a, ω. When a is small, the GOE is small.

GOE in Nonlinear Systems

Questions:

1. Given a nonlinear system, how can one determine if it admits a GOE and for what controls?

1. What control gives the best possible GOE?

2. Are there systems that have (or do not have) a GOE for any admissible control?

GOE in Nonlinear Systems: A Motivating Example

The ribosome flow model (RFM): a phenomenological model for the flow of particles along a 1D chain of n sites. For $n = 3$, the RFM is:

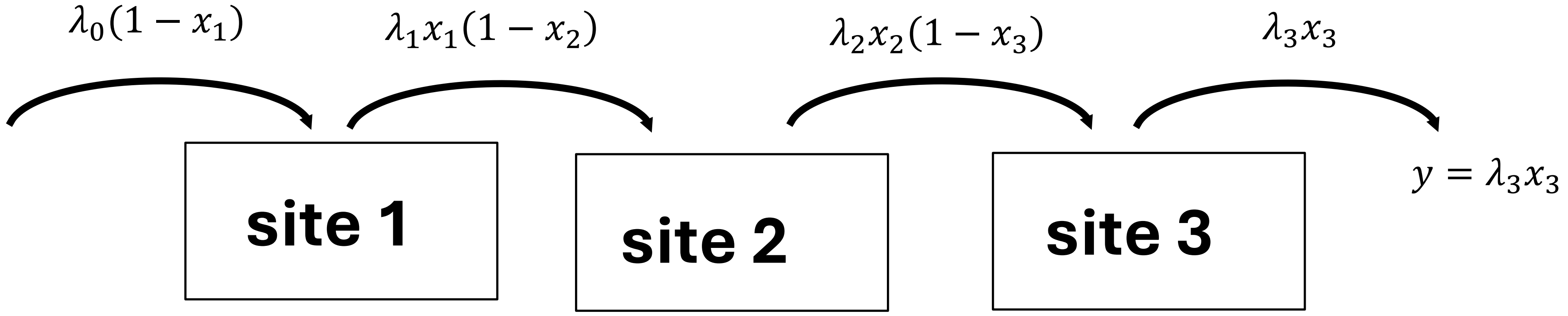
$$\boxed{\begin{aligned}
\dot{x}_1(t) &= \lambda_0(t)\big(1 - x_1(t)\big) - \lambda_1(t)x_1(t)(1 - x_2(t)),\\
\dot{x}_2(t) &= \lambda_1(t)x_1(t)(1 - x_2(t)) - \lambda_2(t)x_2(t)(1 - x_3(t)),\\
\dot{x}_3(t) &= \lambda_2(t)x_2(t)(1 - x_3(t)) - \lambda_3(t)x_3(t),
\end{aligned}}$$

where:

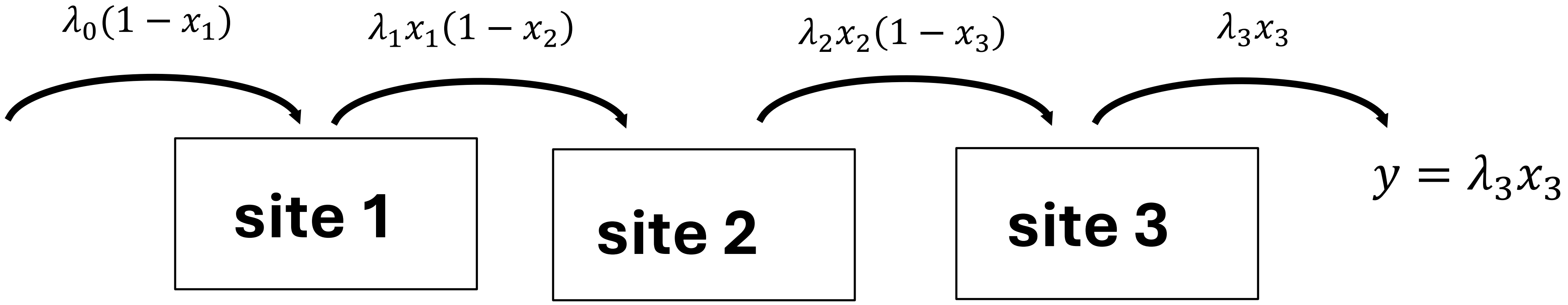
$x_i(t) \in [0,1]$ is the normalized density of particles at site i,
$\lambda_i(t)$ controls the transition rate from site i to site $i + 1$.

GOE in Nonlinear Systems: A Motivating Example

$$
\begin{aligned}
\dot{x}_1(t) &= \lambda_0(t)\big(1 - x_1(t)\big) - \lambda_1(t)x_1(t)(1 - x_2(t)),\\
\dot{x}_2(t) &= \lambda_1(t)x_1(t)(1 - x_2(t)) - \lambda_2(t)x_2(t)(1 - x_3(t)),\\
\dot{x}_3(t) &= \lambda_2(t)x_2(t)\big(1 - x_3(t)\big) - \lambda_3(t)x_3(t),\\
y(t) &= \lambda_3(t)x_3(t).
\end{aligned}
$$



Entrainment in the RFM



If all the $\lambda_i s$ are constant then any solution converges to a steady state (e_1, e_2, e_3).

If all the $\lambda_i s$ are jointly T-periodic then any solution converges to a T$-$perioidic solution ("green wave").

GOE becomes: can periodic traffic lights lead to a better flow, on average?
(I was certain that the answer is YES, but numerical simulations showed NO)

Main Results

We consider SISO nonlinear weakly contractive bilinear control systems.

Such systems entrain:

T-periodic control u

converges to a T-periodic solution γ^u

Main results:

(1) The mapping $u \to \gamma^u(0)$ admits a continuous derivative;

(2) For constant controls, the derivative of $u \to \frac{1}{T}\int_0^T y^u(t)dt$ is zero, implying that GOE is inherently a higher-order phenomenon in the norm of the control perturbation.

Note: the RFM is a nonlinear weakly contractive bilinear control system.

Convexity/Concavity in GOE?

Main results:

(1) The mapping $u \to \gamma^u(0)$ admits a continuous derivative;

(2) For constant controls, the derivative of $u \to \frac{1}{T}\int_0^T \gamma^u(t)dt$ is zero, implying that GOE is inherently a higher-order phenomenon in the norm of the control perturbation.

Question: The GOE depends on a second-order derivative.
When is this always positive or always negative?

This will correspond to systems where periodic controls are always better (worse) than constant controls.

Summary

Entrainment and gain of entrainment (GOE) are important when a system can be controlled either by a constant control or a T-periodic control.

Our results show that GOE is a higher-order phenomenon in the norm of the control perturbation. This leads to many interesting open questions.

ANY QUESTIONS?