LEADER IDENTIFICATION IN SEMI-AUTONOMOUS CONSENSUS PROTOCOLS

CONNECT LAB
Cooperative Networks and Controls

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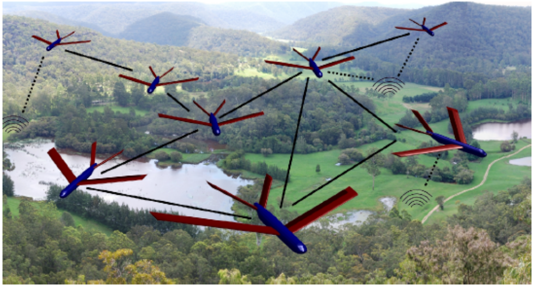
INTRODUCTION

- Multi-agent systems (MAS) consist of autonomous agents interacting to achieve a common goal.
- Their security is vulnerable to cyber-physical attacks, especially through network topology identification.
- **If critical agents are identified**, they become targets for attacks.
- This work explores **identifying leader agents** in networked dynamic systems under a semi-autonomous consensus protocol.



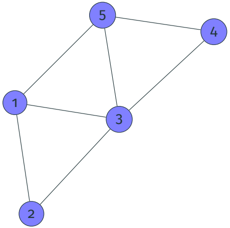
- In the **autonomous consensus** protocol, agents aim to reach agreement via the distributed protocol

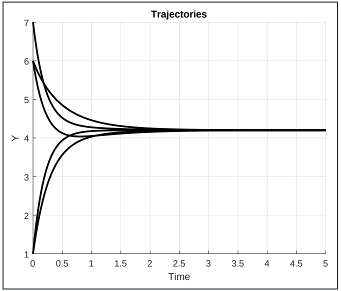
$$\dot{x}_i = \sum_{j \sim i} (x_j - x_i), \quad i \in \mathcal{V}$$

- Under a connectivity assumption of the information exchange graph, the protocol satisfies:

$$\lim_{t \to \infty} x(t) \in \operatorname{span}\{\mathbb{1}_n\}$$

Network Topology $G(\mathcal{V}, \mathcal{E})$

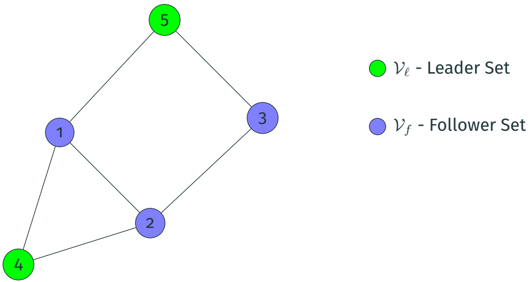


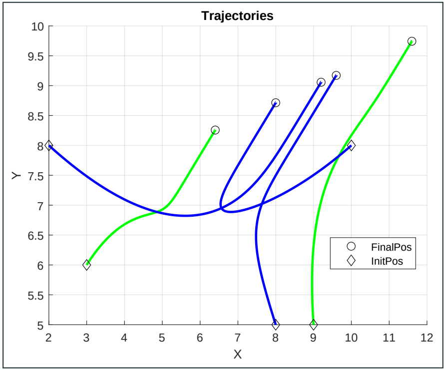


In the **semi-autonomous consensus** protocol, some agents, called leaders, receive an external input:

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

Network Topology



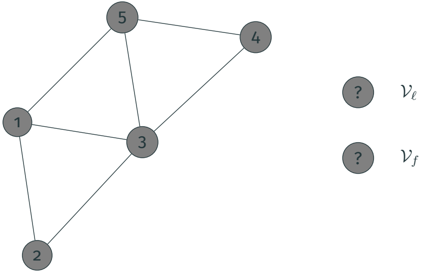


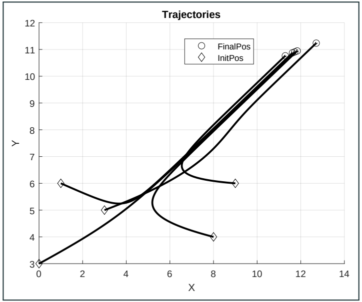
Objective

Identify the leader agents in a semi-autonomous consensus network.

- underlying graph is unknown
- assume constant external inputs
- access to measurements of system state

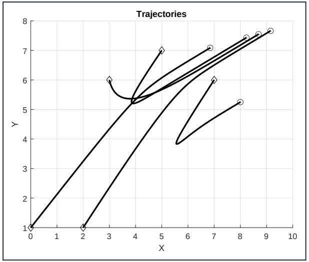
Network Topology



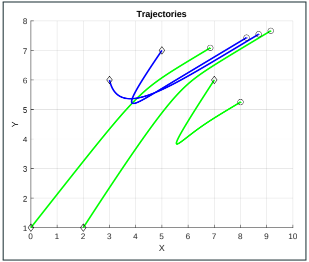


We explore the connection between the Laplacian eigenvectors and leader positions:

- Distributed estimation of Laplacian eigenvectors from system trajectories.
- Identify relationship between eigenvectors of Laplacian with leader positions.



Eigenvector Est. → $\hat{v}$ → Leader Est.



$\mathcal{V}_\ell$ - Leader Set $\mathcal{V}_f$ - Follower Set

FIEDLER EIGENVECTOR

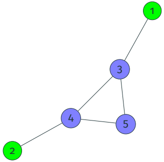
$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

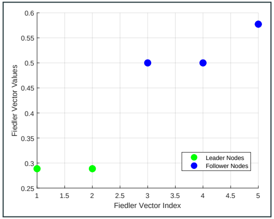
- The semi-autonomous protocol can be written as:

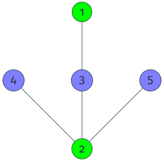
$$\dot{x} = -L_B(\mathcal{G})x + \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0_{|\mathcal{V}_f|\times|\mathcal{V}_\ell|} \end{bmatrix} \begin{bmatrix} u_1^{ex} \\ \vdots \\ u_{|\mathcal{V}_\ell|}^{ex} \end{bmatrix}.$$

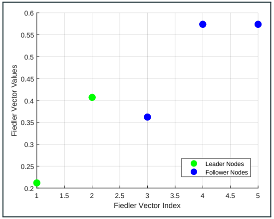
  - $L_B(\mathcal{G})$ is called the grounded Laplacian
  - the eigen-pair (λ_F, v_F) of $L_B(\mathcal{G})$ corresponding to the smallest eigenvalue of are termed the Fiedler eigenvalue and eigenvector

- The Fiedler vector components are associated with the graph structure.

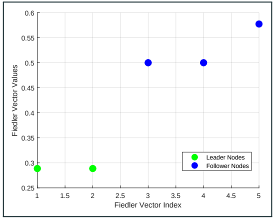


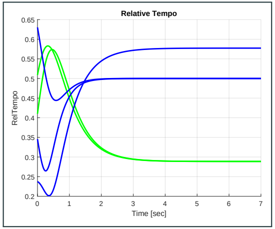






- The velocities of the nodes are linked to the Fiedler vector components.





We will examine a sequence of expanding graphs $\mathcal{G}^{\sigma(i)}$ with some structure constraints:

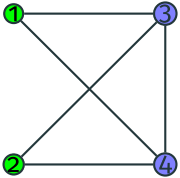
- The leaders set remains constant.
- The leader degree is constant.
- Leader nodes are not connected to each other.

Let $\mathcal{G}_f^{\sigma(i)}$ denote the graph obtained by removing all leader nodes and their incident edges from $\mathcal{G}^{\sigma(i)}$. The additional property in the sequence is as follows:

- The minimum degree in $\mathcal{G}_f^{\sigma(i)}$ is strictly increasing (denoted as $\underline{d}_F$).

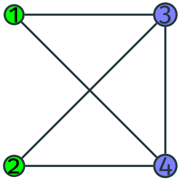
GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

$\underline{d}_F$=1

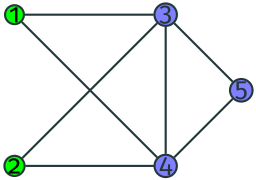


GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

$\underline{d}_F$=1

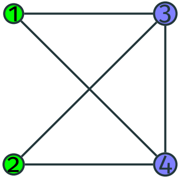


$\underline{d}_F$=2

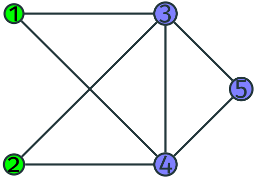


GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

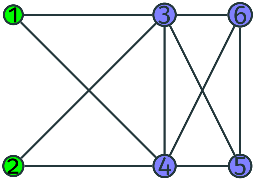
$\underline{d}_F$=1



$\underline{d}_F$=2

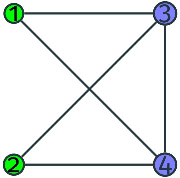


$\underline{d}_F$=3

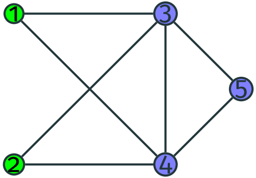


GRAPH SEQUENCE WITH FIEDLER VECTOR SEPARATION

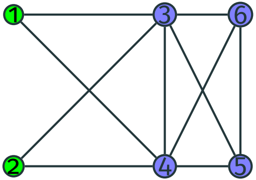
$\underline{d}_F$=1



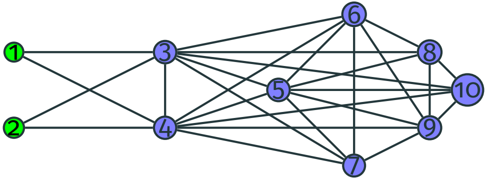
$\underline{d}_F$=2



$\underline{d}_F$=3



$\underline{d}_F$=5



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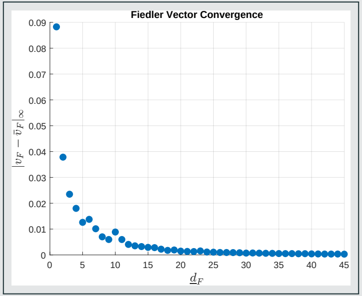
Lemma

The Fidler vector of $\mathcal{G}^{\sigma(i)}$ converges to the following values:

$$\lim_{i\to\infty} [v_F^{\sigma(i)}]_j = [\bar{v}_F]_j \text{ where } [\bar{v}_F]_j = \lim_{i\to\infty} \begin{cases} 1, & j \in \mathcal{V}_f^{\sigma(i)} \\ \frac{\mathbf{d}(j)}{\mathbf{d}(j)+1-\lambda^{\sigma(i)}}, & j \in \mathcal{V}_\ell \end{cases}$$

where $d(j)$ is the node degree and λ is the Fiedler eigenvalue.



Theorem

Let $\mathcal{G}$ be graph where the nodes separated into two groups, leaders $\mathcal{V}_\ell^{\mathcal{G}}$ and followers $\mathcal{V}_f^{\mathcal{G}}$.

If the following conditions are met:

i) $\mathcal{G}$ is connected;
ii) $k \notin \mathcal{N}(j)$ for all $k, j \in \mathcal{V}_\ell$ (leader nodes are not connected to each other);
iii) $1 - \max_{j \in \mathcal{V}_\ell} \frac{d(j)}{d(j)+1-\lambda} > \max_{j,k \in \mathcal{V}_\ell, j>k} |[\bar{v}_{F_s}]_j - [\bar{v}_{F_s}]_k|$
iv) $\underline{d}_F$ is sufficient large,

where λ is the Fiedler eigenvalue of $\mathcal{G}$, then

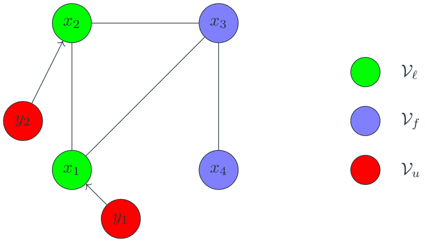
$$\min_{i \in \mathcal{V}_f} [v_F]_i - \max_{i \in \mathcal{V}_\ell} [v_F]_i > \max_{i,j \in \mathcal{V}_\ell, i>j} |[v_{F_s}]_i - [v_{F_s}]_j|.$$

To link the Fiedler vector with node velocities, we transform the semi-autonomous system into an autonomous-like structure:

- Introduce a state variable y representing external control inputs.
- Assume the external input remains constant, giving the dynamics:

$$\dot{y}_i = 0, \quad y_i(0) = u_i^{ex}.$$

Our graph $\bar{\mathcal{G}}$ is directed and consists of three groups: $\mathcal{V}_\ell, \mathcal{V}_f, \mathcal{V}_u$.



- The system dynamics can be expressed as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\bar{L} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} L_B & \bar{L}_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $\bar{L} = L(\bar{\mathcal{G}})$ is the directed graph Laplacian of $\bar{\mathcal{G}}$. The submatrices are given by:

$$L_B = L(\mathcal{G}) + \begin{bmatrix} I_{|\mathcal{V}_\ell|} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_{12} = \begin{bmatrix} I_{|\mathcal{V}_\ell|} \\ 0 \end{bmatrix}.$$

Results

The Fiedler eigenvalue and eigenvector of L_B is also the smallest non-zero eigenvalue for $\bar{L}$.

Lemma

If $\mathcal{G}$ is connected, then the following properties hold for L_B:

- The **Fiedler Eigenvalue** $\lambda_{\mathbf{F}}$ (smallest eigenvalue) of L_B is positive and simple and satisfies

$$0 < \lambda_F \leq 1$$

- The upper bound of the Fiedler eigenvalue is attained iff all nodes in $\mathcal{G}$ are leaders.
- The **Fiedler Eigenvector** v_F is unique (up to scaling) and is the only positive eigenvector.

The relative tempo is the ratio of velocities of agents,

$$[\bar{\tau}(t)]_i = \frac{\dot{x}_i(t)}{\dot{x}_{\text{ref}}(t)}$$

where $\dot{x}_{\text{ref}}$ is the velocity of a specific agent chosen as a common divisor for all others.

For sufficient time T:

$$[\bar{\tau}(t)]_i \simeq [v_F]_i, \quad t > T$$

H. Shao and M. Mesbahi, *Degree of relative influence for consensus-type networks*, Portland, OR, USA, 2014, pp. 2676-2681.

Assuming existence of the result from the theorem, i.e.,

$$\min_{i\in\mathcal{V}_f}[v_F]_i - \max_{i\in\mathcal{V}_\ell}[v_F]_i > \max_{i,j\in\mathcal{V}_\ell, i>j} |[v_{F_s}]_i - [v_{F_s}]_j|.$$

we use the following algorithm to identify the leaders

Algorithm

- Step 1: Measure the agents velocities to an external constant input until steady state.
- Step 2: Calculate the relative tempo and compute the Fiedler vector.
- Step 3: Sort the Fiedler vector $v_{F_s} = \mathrm{sort}(v_F)$ where $[v_{F_s}]_i \leq [v_{F_s}]_{i+1}$.
- Step 4: Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell| = \arg\max_{j\in\{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\}.$$

- Step 5: The leaders are corresponding to the smallest n_l components in v_{F_s}.

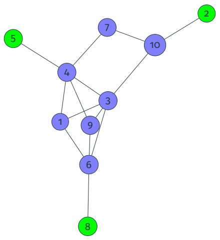
EXAMPLE

In this example, we demonstrate a 2D scenario. We consider a system with $n = 10$ agents, where $\{2, 5, 8\} \in \mathcal{V}_L$. Recall the protocol dynamics:

$$\dot{x}_i = \begin{cases} \sum_{j\sim i}(x_j - x_i) + (u_i^{\text{ex}} - x_i), & i \in \mathcal{V}_\ell, \\ \sum_{j\sim i}(x_j - x_i), & i \in \mathcal{V}_f. \end{cases}$$

The external input provided to the leaders is

$$u = \begin{bmatrix} 40 & 35 & 48 & 44 & 16 & 45 \end{bmatrix}^T$$



EXAMPLE CONT.

The grounded Laplacian and the Fiedler vector is given by:

$$L_B = \begin{bmatrix} 3 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 5 & -1 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 5 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 3 \end{bmatrix}, \quad v_F = \begin{bmatrix} 0.37 \\ \mathbf{0.18} \\ 0.37 \\ 0.35 \\ \mathbf{0.19} \\ 0.34 \\ 0.37 \\ \mathbf{0.19} \\ 0.37 \\ 0.32 \end{bmatrix}$$

EXAMPLE CONT.

Next, we verify the conditions outlined in the Theorem:

- Leaders are not connected to each other.
- Degree distribution condition.
- $\underline{d}_F$ is sufficient large.

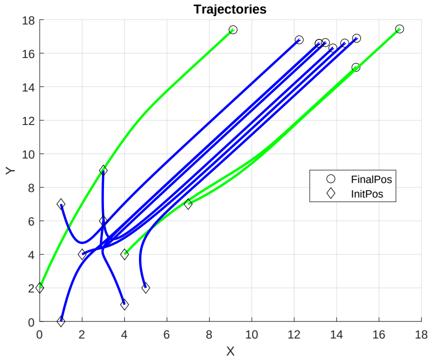
Since all conditions are satisfied, the leaders can be identified using the suggested algorithm.

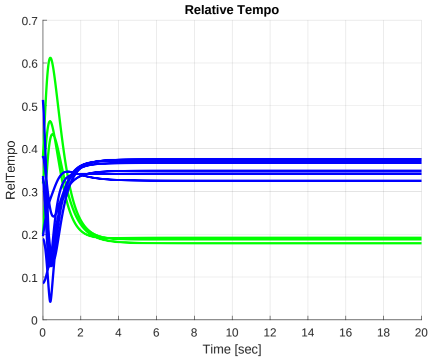
EXAMPLE CONT.

I. Measure the velocities and calculate the relative tempo:

$$\tau = \begin{bmatrix} 0.37 & 0.18 & 0.37 & 0.35 & 0.19 & 0.34 & 0.37 & 0.19 & 0.37 & 0.32 \end{bmatrix}^T$$

We note that this is equal to the Fiedler vector v_F.





II. Identify Leaders

- Sort the Fiedler vector $v_{F_s} = \mathrm{sort}(v_F)$ where $[v_{F_s}]_i \leq [v_{F_s}]_{i+1}$:

$$v_{F_s} = \begin{bmatrix} 0.18 & 0.19 & 0.19 & 0.32 & 0.34 & 0.35 & 0.37 & 0.37 & 0.37 & 0.37 \end{bmatrix}^T$$

$$\text{Index} = \begin{bmatrix} 2 & 5 & 8 & 10 & 6 & 4 & 7 & 9 & 3 & 1 \end{bmatrix}^T$$

- Calculate the number of leaders n_l with

$$n_l = |\mathcal{V}_\ell = \arg \max_{j \in \{1,2,3,\cdots,n-1\}} \{[v_{F_s}]_{j+1} - [v_{F_s}]_j\} = 3.$$

- The leaders correspond to the smallest n_l components in v_{F_s}:

$$\mathcal{V}_\ell = \{2, 5, 8\}$$

- Certain graph structures are more likely to be associated with separation in the components of the Fiedler vector.
- Such graphs can facilitate leader identification through external observation in scenarios with constant external input.

FUTURE WORK

- Investigate scenarios involving non-constant external input signals.
- Develop methods for identifying the complete network structure.
- Explore additional graph topologies related to component separation in the Fiedler vector.