Estimation of Multi-Sinusoidal Signal Parameters Using GPEBO Approach

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Problem statement

We consider the measured signal of the following form

$$y(t) = \sum_{i=1}^{n} A_i sin(\omega_i t + \phi_i), \tag{1}$$

where A_i, ω_i and ϕ_i, $i = 1...n$ are **unknown** parameters (n is **known**).

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where A_i, ω_i and ϕ_i, $i = 1...n$ are **unknown** parameters (n is **known**).

The goal is to construct estimates $\hat{y}(t)$, $\hat{A}_i$, $\hat{\omega}_i$ and $\hat{\phi}_i$ such as

$$\lim_{t\to\infty} (\hat{y}(t) - y(t)) = 0, \tag{2}$$

$$\lim_{t\to\infty} (\hat{A}_i - A_i) = 0, \tag{3}$$

$$\lim_{t\to\infty} (\hat{\omega}_i - \omega_i) = 0, \tag{4}$$

$$\lim_{t\to\infty} (\hat{\phi}_i - \phi_i) = 0. \tag{5}$$

Problem statement

The parameters estimation of sinusoidal signals is an important and classical problem of control theory and could be used in the different areas such as:

1. power quality monitoring;
2. vibration control;
3. periodic disturbance rejection.

Problem statement

Currently, researchers are considering several options for tasks of parameters estimation:

1. pure sinusoidal signal and multisinusoidal signal with time invariant parameters;
2. sinusoidal signals with time varying parameters (amplitude, frequency);
3. noisless sinusoidal signals and signals corrupted by noise.

Problem statement

Methods dedicated to the problem of unknown parameters estimation of the sinusoidal signal:

1. Fourier Transform (FT) Method;
2. Kalman Filters (KF);
3. Adaptive Notch-Filtering (ANF);
4. Magnitude/Phase-Locked Loop approach (MLL/PLL);
5. Dynamic Regressor Extension and Mixing (DREM).

GPEBO approach.

Generalized parameter estimation-based observers (**GPEBO**) is a method of adaptive observers syntesis which was proposed in the IFAC journal *Automatica* [1].

The main feature of GPEBO is that the problem of state observation of dinamical system is recasted as a problem of parameter estimation, namely of the systems initial condition.

[1] R. Ortega, A. Bobtsov, N. Nikolaev, J. Schiffer, and D. Dochain. Generalized parameter estimation-based observers: Application to power systems and chemical-biological reactors. Automatica, vol. 129, p. 109635, 2021.

GPEBO approach.

Based on the GPEBO approach we solve the problem of unknown parameters estimation of the sinusoidal signal using several steps:

1. Special parametrization of the multi-sinusoidal signal like state space linear system (exosystem) with unknown parameters
2. GPEBO observer design
3. Unknown parameters estimation
4. Calculation of the unknown parameters and signal reconstruction

Main result

Consider the measured sinusoidal signal of the form

$$y(t) = A_0 \sin(\omega_0 t + \phi_0), \tag{6}$$

where A_0, ω_0 and ϕ_0 are unknown constant parameters.
There are many approaches which allow us to find unknown parameters. But here we consider how we can apply GPEBO approach.

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Case 1 - How can we find unknown frequency ω_0?
It is well known that (6) can be rewritten in the following form

$$\ddot{y}(t) = -\omega_0^2 y(t). \tag{7}$$

Let us apply LTI filter $\frac{\lambda^2}{(p+\lambda)^2}$. So we can get linear regression model

$$y_e(t) = m^\top(t)\theta, \tag{8}$$

where $y_e(t) = \frac{p^2\lambda^2}{(p+\lambda)^2} y(t)$, $m(t) = \frac{\lambda^2}{(p+\lambda)^2} y(t)$ and $\theta = -\omega_0^2$.
And for this linear regression model we can apply known methods of unknown parameter estimation.

Main result

Case 2 - Consider another parametrization of sinusoidal signal (6)

$$\dot{y}(t) = -A_0\omega(\cos(\omega_0 t + \phi_0)), \tag{9}$$

$$\ddot{y}(t) = -A_0\omega^2 \sin(\omega_0 t + \phi_0) = -\omega^2 y(t). \tag{10}$$

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We can write state space realization of $y(t)$ using the following notations $x_1(t) = (t)y$, $x_2(t) = \dot{x}_1(t) = \dot{y}(t)$,
$\dot{x}_2(t) = \ddot{x}_1(t) = \ddot{y}(t) = -\omega_0^2 y(t) = -\omega_0^2 x_1(t)$,

$$\dot{x}(t) = Ax(t); x(0), \tag{11}$$

$$y(t) = C^\top x, \tag{12}$$

where $A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So we made the **step 1** of our approach - we get special parametrization of initial signal.

For state-space realization of $y(t)$ we can apply GPEBO approach.

Main result

GPEBO realization - step 2.
Let us consider a dynamical system

$$\dot{\xi}(t) = A\xi(t); \xi(0) = 0, \tag{13}$$

and an error signal in the following form

$$e(t) = x(t) - \xi(t). \tag{14}$$

Then for derivative of $e(t)$ we have

$$\dot{e}(t) = \dot{x}(t) - \dot{\xi}(t) = Ax(t) - A\xi(t) = Ae(t), \tag{15}$$

and for solution of $e(t)$

$$e(t) = \Phi(t)\theta, \tag{16}$$

where $\Phi(t)$ is the transition matrix and $\theta = e(0)$ are initial conditions.
Now we can rewrite $y(t)$ in the following form

$$y(t) = C^\top x(t) = C^\top \xi(t) - C^\top \Phi(t)\theta. \tag{17}$$

Main result

From previous equation we can find a linear regression equation

$$y_e(t) = m^\top(t)\theta, \tag{18}$$

where $y_e(t) = y(t) - C^\top\xi(t)$ and $m(t) = C^\top\Phi$.

So we can estimate unknown parameters θ using known methods (**step 3**).

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We can calculate the unknown paprameters of the sinusoidal signal $y(t)$ and reconstruct its form (**step 4**) using equations

$$y(t) = A_0\sin(\omega_0 t + \phi_0) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t), \tag{19}$$

$$c_1 = y(0), \quad c_2 = \frac{\dot{y}(0)}{\omega_0}, \tag{20}$$

$$A_0 = \sqrt{c_1^2 + c_2^2}, \phi_0 = \arctan\left(\frac{c_1}{c_2}\right), \tag{21}$$

where $y(0)$ and $\dot{y}(0)$ are the initial conditions

$$\hat{y}(t) = C^\top\xi(t) - C^\top\Phi(t)\hat{\theta}. \tag{22}$$

But we **need to know the frequency**, so we cannot find all signal parameters.

Case 3 - If we want to find all parameters we can use another parametrization of $y(t)$.
Let us consider a multisinusoidal signal of the form

$$y(t) = \sum_{i=1}^{n} A_i sin(\omega_i t + \phi_i), \tag{23}$$

where A_i, ω_i and ϕ_i, $i = 1...n$ are **unknown** parameters (n is **known**).
Signal (23) can be represented as a solution of a differential equation of the view

$$(p^2 - \theta_1)(p^2 - \theta_2)...(p^2 - \theta_n)y(t) = 0, \tag{24}$$

where $\theta_i = -\omega_i^2$, $i = 1...n$.

Main result

We can rewrite previous equation in the following form

$$y(t) = \theta_1^* \frac{y(t)}{p^2} + \theta_2^* \frac{y(t)}{p^4} + ... + \theta_n^* \frac{y(t)}{p^{2n}}, \tag{25}$$

where for unknown parameters we have

$$\begin{aligned}
\theta_1^* &= \theta_1 + \theta_2 + ...\theta_n, \\
\theta_2^* &= -\theta_1\theta_2 - \theta_1\theta_3 - ... - \theta_{n-1}\theta_n, \\
&\vdots \\
\theta_n^* &= (-1)^{n+1}\theta_1\theta_2...\theta_n.
\end{aligned}$$

And we can write state space realization of $y(t)$.

Main result

For state space realization of $y(t)$ we have

$$\dot{x}(t) = Ax(t) + By(t),\ x(0), \tag{26}$$

$$\hat{y}(t) = C^\top x(t), \tag{27}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{2nx2n}$, $B = \begin{bmatrix} 0 \\ \theta_1^* \\ 0 \\ \vdots \\ \theta_n^* \end{bmatrix}_{2nx1}$, $C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2nx1}$,

$x(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(2n-1)}(0) \end{bmatrix}_{2nx1}$.

Main result

Let us consider case for $n = 4$. For this case we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \theta_1^* \\ 0 \\ \theta_2^* \\ 0 \\ \theta_3^* \\ 0 \\ \theta_4^* \end{bmatrix}, \; C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \theta_1^* &= (\theta_1 + \theta_2 + \theta_3 + \theta_3), \\ \theta_2^* &= -(\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4), \\ \theta_3^* &= (\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4), \\ \theta_4^* &= -(\theta_1\theta_2\theta_3\theta_4). \end{aligned}$$

We can try to use GPEBO approach, but for this case we will have a problem - matrix A is unstable and we will have unbounded growing of matrix exponential e^{At} so we need to transform our approach.

Main result

Consider matrix

$$A_0 = A - LC^\top, \tag{28}$$

where vector L is such that matrix A_0 is stable.

So we transform our parametrization model to the following form

$$\dot{x}(t) = A_0 x(t) + By(t) + Ly(t), \; x(0), \tag{29}$$

$$\hat{y}(t) = C^\top x(t), \tag{30}$$

where vector L is choosen so that the matrix A_0 is stable (Hurwitz) and now we need to find the observer for LTI system with unknown parameters.

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Consider GPEBO observer of the following form

$$\dot{\xi}(t) = A_0 \xi + Ly(t),\ \xi(0) = 0, \tag{31}$$

$$\dot{\eta}(t) = A_0 \eta(t) + Iy(t),\ \eta(0) = 0, \tag{32}$$

$$\dot{\Phi}(t) = A_0 \Phi(t),\ \Phi(0) = I, \tag{33}$$

and an error equation of the following form

$$e(t) = x(t) - \xi(t) - \eta(t)B. \tag{34}$$

Main result

For derivative of $e(t)$ we have

$$\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\xi}(t) - \dot{\eta}(t)B \\
&= A_0x(t) + By(t) - A_0\xi - Ly(t) - (A_0\eta(t) + Iy(t))B \\
&= A_0x(t) + LC^\top x(t) + By(t) - A_0\xi - Ly(t) - A_0\eta(t)B - Iy(t)B \\
&= A_0e(t), \\
\dot{e}(t) &= A_0e(t). \qquad (35)
\end{aligned}$$

Main result

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&= A_0 x(t) + LC^\top x(t) + By(t) - A_0\xi - Ly(t) - A_0\eta(t)B - Iy(t)B \\
&= A_0 e(t), \\
\dot{e}(t) &= A_0 e(t). \qquad (35)
\end{aligned}
$$

And now we can find a linear regression model

$$
\begin{aligned}
e(t) &= x(t) - \xi(t) - \eta(t)B, \\
C^\top e(t) &= C^\top x(t) - C^\top \xi(t) - C^\top \eta(t)B, \\
C^\top \Phi(t)\theta &= y(t) - C^\top \xi(t) - C^\top \eta(t)B, \\
y_e(t) &= m^\top(t)\theta, \qquad (36)
\end{aligned}
$$

where $y_e(t) = y(t) - C^\top \xi(t)$, $m(t) = [C^\top \eta(t) \;\; C^\top \Phi]$, $\theta = \begin{bmatrix} B \\ x(0) \end{bmatrix}$.

And now we can use known methods which allow us to find unknown parameters of the LRE.

Simulation results - pure sinusoidal signal.

For simulation we used sinusoidal signal

$$y(t) = A_1 sin(\omega_1 t + \phi_1), \tag{37}$$

where $A_1 = 1$, $\omega_1 = 2$, $\phi_1 = \pi/6$.

For this case the parameterized model has the following form

$$\dot{x}(t) = A_0 x(t) + By(t) + Ly(t),\ x(0), \tag{38}$$

$$\hat{y}(t) = C^\top x(t), \tag{39}$$

with $A_0 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \theta_1^* \end{bmatrix}$, $\theta_1^* = -\omega_1^2$, $L = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0.5 \\ 1.7321 \end{bmatrix}$.

For parameters estimation we used Least-Squares algorithm with Forgetting Factor (LS+FF) [2].

$$\dot{\hat{\theta}} = \alpha F m^\top (y - m\hat{\theta}),\ \hat{\theta}(0) =: \theta_0, \tag{40a}$$

$$\dot{F} = \begin{cases} -\alpha F m^\top m F + \beta F & \text{if } \|F\| \leq M \\ 0 & \text{otherwise} \end{cases}, \tag{40b}$$

with $F(0) = \frac{1}{f_0} I$ and tuning gains $\alpha > 0,\ f_0 > 0,\ \beta \geq 0$ and $M > 0$.

[2] Ioannou P. and J. Sun 'Robust Adaptive Control' published by Prentice Hall, Inc in 1996

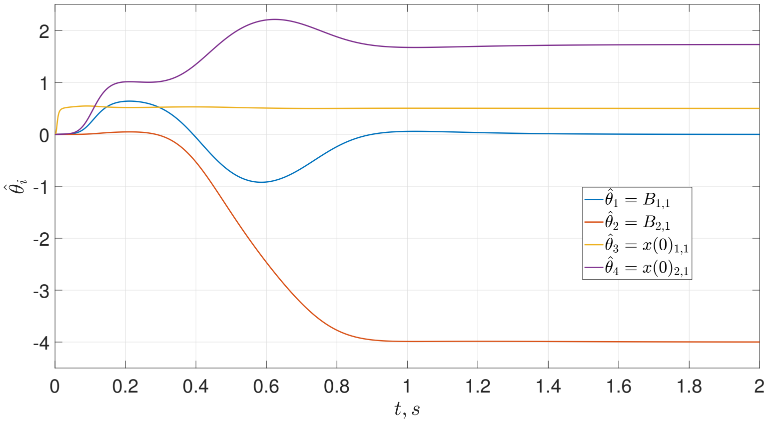


Figure 1: Transients of unknown parameters estimations $\hat{\theta}_i(t)$

We can calculate the unknown parameters using equation

$$y(t) = A_1 \sin(\omega_1 t + \phi_1) = c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t), \tag{41}$$

$$c_1 = y(0), \quad c_2 = \frac{\dot{y}(0)}{\omega_1}, \tag{42}$$

$$A_1 = \sqrt{c_1^2 + c_2^2}, \phi_1 = \arctan\left(\frac{c_1}{c_2}\right), \tag{43}$$

where $\hat{\omega}_1 = \sqrt{\mid \hat{\theta}_2 \mid} = 2$, $y(0) = \hat{\theta}_3 = 0.5$ and $\dot{y}(0) = \hat{\theta}_3 = 1.732$. So we have

$$\hat{\omega}_1 = \sqrt{\mid \hat{\theta}_2 \mid} = \sqrt{\mid -4 \mid} = 2, \tag{44}$$

$$\hat{c}_1 = 0.5, \quad \hat{c}_2 = \frac{1.732}{2} = 0.866, \tag{45}$$

$$\hat{A}_1 = \sqrt{\hat{c}_1^2 + \hat{c}_2^2} = \sqrt{0.5^2 + 0.866^2} = 1, \tag{46}$$

$$\hat{\phi}_1 = \arctan\left(\frac{\hat{c}_1}{\hat{c}_2}\right) = \arctan\left(\frac{0.5}{0.866}\right) = 0.5236 \tag{47}$$

For this case we had real values $A_1 = 1$, $\omega_1 = 2$, $\phi_1 = \pi/6$.

And also we can reconstruct the measured signal $y(t)$ using equation

$$\hat{y}(t) = C^\top \xi(t) - C^\top \Phi(t)\hat{\theta}. \tag{48}$$

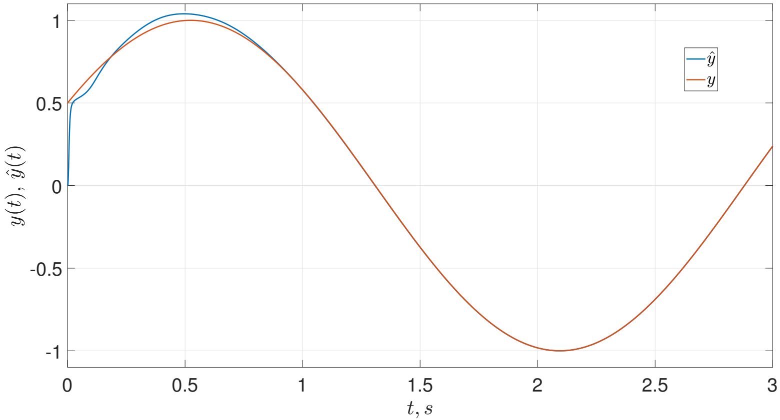


Figure 2: Transients of $y(t)$ and $\hat{y}(t)$

Simulation results - multisinusoidal signal.

For simulation we used multisinusoidal signal in the following form

$$y(t) = \sum_{i=1}^{4} A_i sin(\omega_i t + \phi_i), \tag{49}$$

where $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, $\omega_4 = 4$, $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\phi_3 = 0.3$ and $\phi_4 = 0.4$.

For parameters estimation we used LS+FF algorithm.

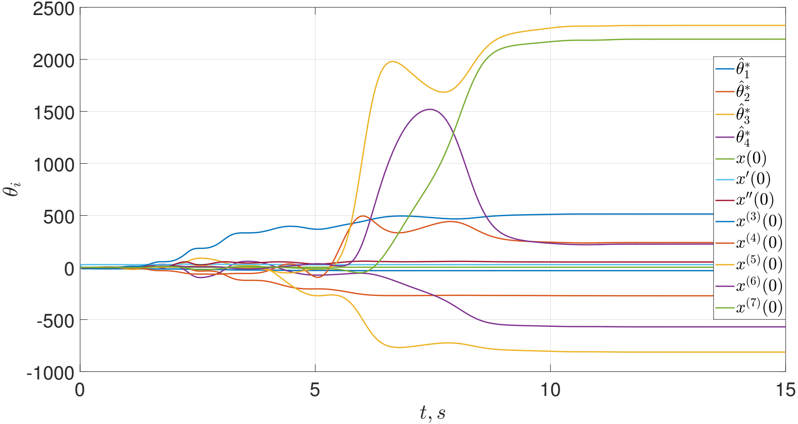


Figure 3: Transients of unknown parameters estimations $\hat{\theta}_i(t)$

For parameters estimation we used LS+FF algorithm.

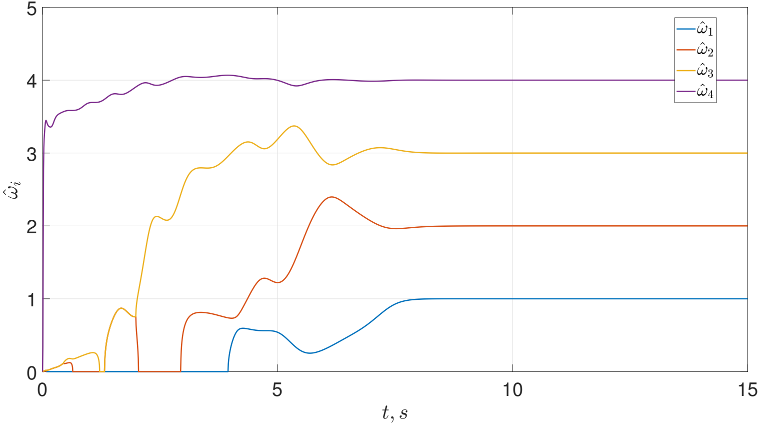
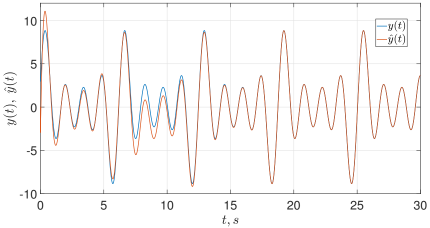
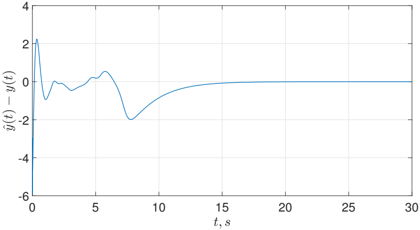


Figure 4: Transients of unknown frequences estimations $\hat{\omega}_i(t)$



a)



b)

Figure 5: Transients for $y(t)$ and $\hat{y}(t)$ a) and for error $e(t)$ b).

Thank you for your attention!