Gramian-Based Analysis of Parametric Uncertainties

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Trajectory sensitivity model for a system with uncertain parameters

Consider a continuous-time system

$$\dot{x}(t,q) = A(q)x(t,q) + B(q)u(t); \tag{1}$$

$$y(t,q) = C(q)x(t,q), \tag{2}$$

where $x(t,q) \in R^n$ is state, $y(t,q) \in R^m$ is output, $u(t) \in R^r$ is control;

$A(q) \in R^{n\times n}, B(q) \in R^{n\times r}, C(q) \in R^{m\times n};$

q is vector of quasy-stationary parameters $(\dot{q}(t) = 0)$, which causes variation $q = q_0 + \Delta q, q \in R^p$, q_0 are nominal values of parameters.

Trajectory sensitivity model for a system with uncertain parameters

Let us write state and output response of the system for arbitrary value of parameters vector q:

$$x(t, q = q_0 + \Delta q) = x(t) + \Delta x(t, q_0, \Delta q), \tag{3}$$

$$y(t, q = q_0 + \Delta q) = y(t) + \Delta y(t, q_0, \Delta q), \tag{4}$$

where $x(t) = x(t, q_0), y(t) = y(t, q_0)$; $x(t), y(t)$ are state and output nominal trajectories of the system, $\Delta x(t, q_0, \Delta q)$, $\Delta y(t, q_0, \Delta q)$ are additional state and output responses caused by parameters variation.

Trajectory sensitivity model for a system with uncertain parameters

Let us assume that norm of parameters vector variation is small $x(t,q)$ and $y(t,q)$ are continuously differentiable with respect to q. Then (3) and (4) can be written as:

$$x(t,q) = x(t) + \left.\frac{\partial x(t,q)}{\partial q}\right|_{q=q_0} \Delta q + O_x^2(\Delta q), \tag{5}$$

$$y(t,q) = y(t) + \left.\frac{\partial y(t,q)}{\partial q}\right|_{q=q_0} \Delta q + O_y^2(\Delta q), \tag{6}$$

where the following relations hold

$$\lim_{\|\Delta q\| \to 0} \frac{O_x^2(\Delta q)}{\|\Delta q\|} = 0, \quad \lim_{\|\Delta q\| \to 0} \frac{O_y^2(\Delta q)}{\|\Delta q\|} = 0. \tag{7}$$

Trajectory sensitivity model for a system with uncertain parameters

If we use (3)-(7) then for additional responses $\Delta x(t, q_0, \Delta q)$, $\Delta y(t, q_0, \Delta q)$ of parametrically disturbed system is possible to write

$$\Delta x(t, q_0, \Delta q) = \Sigma(t)\Delta q, \tag{8}$$

$$\Delta y(t, q_0, \Delta q) = \Pi(t)\Delta q, \tag{9}$$

where matrices of state and output trajectory sensitivity $\Sigma(t)$ and $\Pi(t)$ take the view

$$\Sigma(t) = row\left\{\sigma_j(t) = \left.\frac{\partial x(t,q)}{\partial q_j}\right|_{q=q_0}, j = \overline{1, p}\right\}, \tag{10}$$

$$\Pi(t) = row\left\{\eta_j(t) = \left.\frac{\partial y(t,q)}{\partial q_j}\right|_{q=q_0}, j = \overline{1, p}\right\}. \tag{11}$$

Trajectory sensitivity model for a system with uncertain parameters

$\sigma_j(t)$ and $\eta_j(t)$ are 1st order state and output trajectory sensitivity functions to variations of j-th entry q_j of vector q:

$$\sigma_j = \frac{\partial x_i(t,q)}{\partial q_j}, i = \overline{1,n}, \tag{12}$$

$$\eta_j(t) = \frac{\partial y_k(t,q)}{\partial q_j}, k = \overline{1,m}. \tag{13}$$

Trajectory sensitivity model for a system with uncertain parameters

Let us develop trajectory sensitivity model for the system (1), (2)

$$\dot{x}(t,q) = A(q)x(t,q) + B(q)u(t);\; y(t,q) = C(q)x(t,q),$$

Let us differentiate (1), (2) with respect to q_j at $q = q_0$:

$$\frac{\partial}{\partial q_j}(\dot{x}(t,q))\Big|_{q=q_0} = \frac{\partial}{\partial q_j}\left(\frac{dx(t,q)}{dt}\right)\Big|_{q=q_0} = \frac{\partial}{\partial t}\left(\frac{dx(t,q)}{dq_j}\right)\Big|_{q=q_0} = \dot{\sigma}_j(t), \qquad (14)$$

and also introduce the notations

$$A_{q_j} = \frac{\partial A(q)}{\partial q_j}\Big|_{q=q_0};\; B_{q_j} = \frac{\partial B(q)}{\partial q_j}\Big|_{q=q_0};\; C_{q_j} = \frac{\partial C(q)}{\partial q_j}\Big|_{q=q_0}, \qquad A(q)|_{q=q_0} = A,$$

$$B(q)|_{q=q_0} = B,\;\; C(q)|_{q=q_0} = C,\;\; x(t,q)|_{q=q_0} = x(t),\;\; y(t,q)|_{q=q_0} = y(t). \qquad (15)$$

Trajectory sensitivity model for a system with uncertain parameters

For j –th trajectory sensitivity model (TSM) we have:

$$\dot{\sigma}_j(t) = A\sigma_j(t) + A_{\sigma_j}x(t) + B_{q_j}u(t); \tag{16}$$

$$\eta_j(t) = C\sigma_j(t) + C_{q_j}x(t). \tag{17}$$

TSM (18), (19) generates state $\sigma_j(t)$ and output $\eta_j(t)$ trajectory sensitivity functions if it is supplemented with model of the nominal system (1), (2) for $q = q_0$:

$$\dot{x}(t) = Ax(t) + Bu(t); \tag{18}$$

$$y(t) = Cx(t). \tag{19}$$

Trajectory sensitivity model for a system with uncertain parameters

Let us rewrite model (16)-(19) as an augmented system with state vector $\tilde{x}_j = col\{x, \sigma_j\}, \tilde{x}_j \in R^{2n}$

$$\dot{\tilde{x}}_j(t) = \tilde{A}_j \tilde{x}_j(t) + \tilde{B}_j u(t);\ \tilde{x}_j(0) = col\{x(0), 0\}, \tag{20}$$

$$x(t) = \tilde{C}_{xj}\tilde{x}_j(t); y(t) = \tilde{C}_j \tilde{x}_j(t); \sigma_j(t) = \tilde{C}_{\sigma j}\tilde{x}_j(t); \eta_j(t) = \tilde{C}_{\eta j}\tilde{x}_j(t), \tag{21}$$

where

$$\tilde{A}_j = \begin{bmatrix} A & 0 \\ A_{q_j} & A \end{bmatrix}; \tilde{B}_j = \begin{bmatrix} B \\ B_{q_j} \end{bmatrix}; \tilde{C}_{xj} = [I_{n\times n} \quad 0_{n\times n}];\ \tilde{C}_j = [C \quad 0_{m\times n}];$$

$$\tilde{C}_{\sigma j} = [0_{n\times n} \quad I_{n\times n}];\ \tilde{C}_{\eta j} = \begin{bmatrix} C_{q_j} & C \end{bmatrix}.$$

This model will be used to analyze uncertainties influence on the output of the system (1), (2).

Gramians

Consider a linear system

$$\dot{x}(t) = Fx(t) + Gu(t); \tag{22}$$

$$y(t) = Nx(t). \tag{23}$$

For systems of the view (22), (23) controllability Gramians are defined as follows

$$W_c(t) = \int_0^t e^{F\tau} G G^T e^{F^T \tau} d\tau, \tag{24}$$

If system (22), (23) is stable, Gramians are calculated according to Lyapunov equations

$$W_c F^T + F W_c = -G G^T, \qquad W_o F + F^T W_o = -N^T N. \tag{25}$$

Main Result

Consider models (20), (21), developed for each of the parametric uncertainties and evaluate the contribution of each of the uncertainties to the output variable of the system (1), (2). Let us calculate controllability Gramians for the channels "control $u(t)$ – output $y(t)$" (20), (21) and "control $u(t)$ – output of j-th TSM $\eta_j(t)$":

$$W_c\colon\ AW_c + W_cA^T = -BB^T, \tag{26}$$

$$\widetilde{W}_{cj}\colon\ \tilde{A}_j\widetilde{W}_{cj} + \widetilde{W}_{cj}\tilde{A}_j^T = -\tilde{B}_j\tilde{B}_j^{\,T}, \tag{27}$$

And output Gramians for these channels:

$$W_{yc}\colon\ W_{yc} = CW_cC^T; \tag{28}$$

$$\widetilde{W}_{ycj}\colon\ \widetilde{W}_{ycj} = \tilde{C}_{\eta j}\widetilde{W}_{cj}\tilde{C}_{\eta j}^{\,T}. \tag{29}$$

Main Result

Let us use Gramians (28), (29) provided that $u(t)$ belongs to the unit sphere. Let us form quadratic forms on the above Grammians $u^T\widetilde{W}_{ycj}u$ and $u^T W_c u$ and construct on their basis a generalized Rayleigh relation for which the following estimating inequality holds:

$$\mu_{min} \leq \frac{u^T\widetilde{W}_{ycj}u}{u^T W_c u} \leq \mu_{max}. \quad (30)$$

In (30) constants μ_{min}, μ_{max} are extreme elements of the spectrum of solutions of the generalized characteristic equation

$$det\left(\widetilde{W}_{ycj} - \mu_j W_c\right) = 0. \quad (31)$$

Main Result

Let us write the algorithm to compare uncertainties by level of influence on the system output.

Step 1. Describe the system in the form (1),(2).

Step 2. Contstrct trajectory sensitivity model for each uncertainty q_j, $j = 1, \dots p$.

Step 3. Calculate controllability Gramians (28), (29).

Step 4. Solve equation (31).

Step 5. Based on the solution of equation (31), determine which uncertainties most strongly influence the system output.

Example

Let us consider a system with parametric uncertainties described by equations (1), (2).

Step 1. State, control and output matrices are

$$A(q) = \begin{bmatrix} -0.7(1+q_1) & -0.5(1+q_2) \\ 1 & -0.2(1+q_3) \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Vector of parameters $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, |q_j| \leq 1$, nominal values $q_{j0} = 0$.

Example

Step 2. Let us construct models of trajectory sensitivity of the form (18), (19) for each uncertainty. Then the matrices describing them take the form

$$\tilde{A}_1 = \begin{bmatrix} A & 0 \\ A_{q_1} & A \end{bmatrix};\ \tilde{B}_1 = \begin{bmatrix} B \\ B_{q_1} \end{bmatrix};\ \tilde{C}_{\eta_1} = [C_{q_1} \quad C], A = \begin{bmatrix} -0.7 & -0.5 \\ 1 & -0.2 \end{bmatrix},$$

$$A_{q_1} = \begin{bmatrix} -0.7 & 1 \\ 1 & 0 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; C_{q_1} = [0 \quad 0].$$

$$\tilde{A}_2 = \begin{bmatrix} A & 0 \\ A_{q_2} & A \end{bmatrix};\ \tilde{B}_2 = \begin{bmatrix} B \\ B_{q_2} \end{bmatrix};\ \tilde{C}_{\eta_2} = [C_{q_2} \quad C],$$

$$A_{q_2} = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}, B_{q_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; C_{q_2} = [0 \quad 0].$$

$$\tilde{A}_3 = \begin{bmatrix} A & 0 \\ A_{q_3} & A \end{bmatrix};\ \tilde{B}_3 = \begin{bmatrix} B \\ B_{q_3} \end{bmatrix};\ \tilde{C}_{\eta_3} = [C_{q_3} \quad C],$$

$$A_{q_3} = \begin{bmatrix} 0 & 0 \\ 1 & -0.2 \end{bmatrix}, B_{q_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_{q_3} = [0 \quad 0].$$

Example

Step 3. Let's calculate the controllability Gramians (30), (31):

$$W_{yc} = 0.4514, \widetilde{W}_{yc1} = 0.0708, \widetilde{W}_{yc2} = 0.3858, \widetilde{W}_{yc3} = 0.0911.$$

Step 4. Solving equation (31) we have $\mu_1 = 0.1569$, $\mu_2 = 0.3547$, $\mu_3 = 0.2018$.

Step 5. Based on the solutions of equation (31), we conclude that the output of the system with parametric uncertainties will be most strongly influenced by the uncertainty q_2.

Thank you for your attention!