Utilization of Noise for the Control of a class of Non-Linear Systems

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Outline

Introduction

- Deterministic state-feedback controllers are far more popular than stochastic control achieving Stochastic Antiresonance (SAR), where state-multiplicative noise is applied to stabilize non linear systems.
- The reverse phenomenon, Stochastic Resonance (SR), has been mentioned in connection with several topics such as periodic occurrence of ice ages, animal behavior, sensory neurons and ionic channels, optical systems and so on.
- SAR in squid giant axons has been studied in [Borkowsky 2010], where the potential for therapeutic neurological applications was pointed out (i.e. neuromodulation).

Introduction (contd.)

- Recently the stabilization of a class of non linear systems with sector bounded non linearities has been analyzed, where Linear Matrix Inequalities (LMI) based condition have been derived in [SY24] (Stoica & Yaesh 2024). *Current talk objectives are :*
- Presenting those LMI-based conditions and their application.
- Expanding SAR applications using the universal approximation theorem [Cybenko 1989] for systems that are not apriori modeled with sector bounded uncertainties.

Preliminaries

- In the present talk we deal with stochastic systems, involving state multiplicative noise. These systems can be represented as Itô type stochastic differential equations (SDE) as follows:

$$dx = f(x(t), t)dt + g(x(t), t)d\beta(t)$$

where $\beta(t)$ is a Wiener process with $Ed\beta^2(t) = Q(t)dt$.

- To calculate the differential of a scalar valued function $\Phi(x)$, we use Itô formula

$$d\phi = \phi_t dt + \Phi_x^T dx + \frac{1}{2}TrgQg^T\phi_{xx}dt = \mathcal{L}\phi + Md\beta$$

Preliminaries

- We denoted the infinitesimal generator by $\mathcal{L}\phi := \phi_t dt + \phi_x^T f(x(t),t)dt + \frac{1}{2}TrgQg^T\phi_{xx}dt$.
- We also denoted $M := \Phi_x g$.
- If $\phi(x) = V(x) > 0$ and $\mathcal{L}V < 0$ then V is said to be a Lyapunov function, where $V(x(t)) = V(x_0) + \int_0^t \mathcal{L}V(x(t))dt + \int_0^t M d\beta$
- When $\mathcal{L}V < 0$ for all nonzero x, and $g(x(t),t) = x(t)$, then we get $M = V_x x$ and the origin is asymptotically stable in probability [Khasminskii, 2012].
- Although this type of a stability is weaker than mean square stability, it is still of practical value as we will see in the sequel.

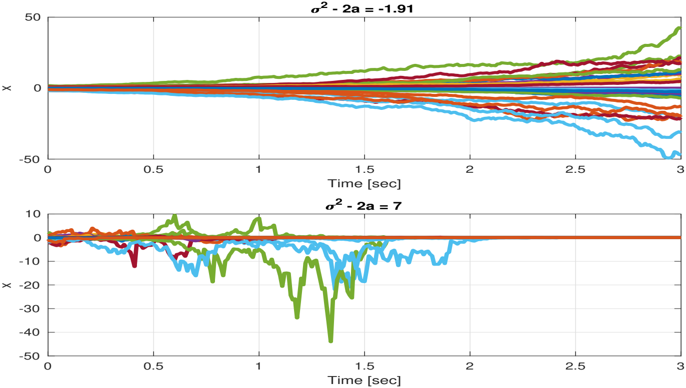
Preliminaries (contd.)

- For motivation and initial feel of SAR consider the following linear continuous-time scalar stochastic system $dx(t) = ax(t)dt + \sigma x(t)d\beta(t)$, where $a > 0$ and where $\beta(t)$ is a Wiener process with $E\{d\beta^2(t)\} = dt$. This system is clearly unstable for $\sigma = 0$.
- However, for $\sigma \neq 0$, we choose the positive definite function $V(x) = |x|^\nu, \nu \in (0,1)$ and calculate

$$\mathcal{L}V = V_x ax + \frac{1}{2}V_{xx}\sigma^2 x^2 = |x|^\nu \nu \left(a + \frac{\sigma^2}{2}(\nu - 1) \right).$$

- Taking $\sigma > 0$ such that $\sigma^2 > 2a$, it follows that for $0 < \nu < 1 - \frac{2a}{\sigma^2}$ we get $\mathcal{L}V < 0 \rightarrow$ asymptotically stable in probability.

Preliminaries (contd.)



Scalar system which is unstable without state-multiplicative noise, 1a) $\sigma^2 < 2a$ SAR not attained, 1b) $\sigma^2 > 2a$ SAR attained.

Problem formulation

Consider the following system:

$$\begin{aligned} dx(t) &= Ax(t)dt + Ff(y(t))dt + Dx(t)d\beta(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ denotes the state vector, $y \in \mathcal{R}^n$ is the measured system output and where $\beta(t) \in \mathcal{R}$ is a standard Wiener process with $E\{d\beta^2(t)\} = dt$ on the given probability space which is also independent of x_0.

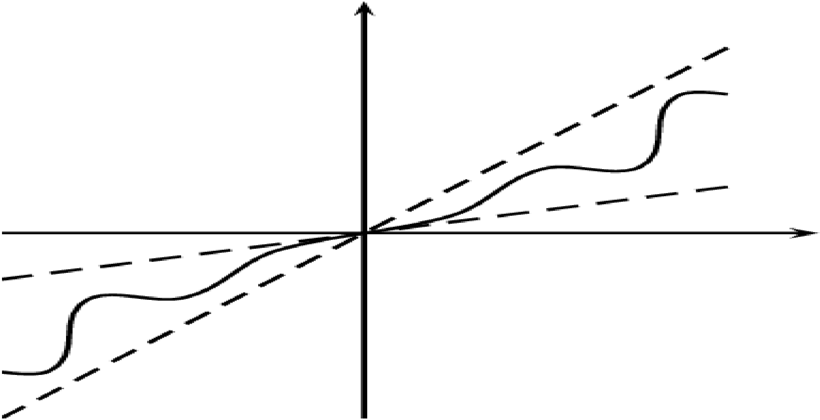
Problem formulation (contd.)

- The elements of y are $y_i = C_i x \in \mathcal{R}$, $i = 1, ..., n$, where C_i is the i'th row vector of C, namely $y_i = \sum_{j=1}^{n} C_{ij} x_j$
- The components $f_i(y_i)$ of $f(y)$ satisfy the sector conditions $0 \le y_i f_i(y_i) \le s_i y_i^2$ which are equivalent to

$$f_i(y_i)(f_i(y_i) - s_i y_i) \le 0,\ i = 1, ..., n. \tag{2}$$

- Define $S = diag\{s_1, s_2, ..., s_n\}$ and assume $\delta_i > 0$, $i = 1, \ldots, n$ such that $\frac{df_i(y_i)}{dy_i} < \delta_i$, and $C^T C = I$.
- Our aim is to provide conditions for stability of this system.

Problem Formulation (contd.)



Sector Bounded Nonlinearities

Problem Solution

The main result in [SY24] states that if there exist $\nu \in (0,1)$, $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq 0$, $i = 1, \ldots, n$ and $\mathcal{T} = diag(\tau_1, \cdots, \tau_n)$, $\tau_i \geq 0$, $i = 1, \ldots, n$ such that

$$\mathcal{N} := \begin{bmatrix} \mathcal{N}_{11}(\nu, \Lambda) & \mathcal{N}_{12}(\nu, \Lambda, \mathcal{T}) \\ \mathcal{N}_{12}^T(\nu, \Lambda, \mathcal{T}) & \mathcal{N}_{22}(\Lambda, \mathcal{T}) \end{bmatrix} < 0 \tag{3}$$

where $\Delta := diag\left(\delta_1, \ldots, \delta_n\right)$ and where

$$\begin{aligned} \mathcal{N}_{11}(\nu, \Lambda) &:= \nu\left[A^T + A - \sigma^2(1-\nu)I\right] + \sigma^2 C^T \Lambda \Delta C \\ \mathcal{N}_{12}(\nu, \Lambda, \mathcal{T}) &:= \nu F + \left[A - \frac{\sigma^2}{2}\left(1 - \frac{\nu}{2}\right) I\right]^T C^T \Lambda + S C^T \mathcal{T} \\ \mathcal{N}_{22}(\Lambda, \mathcal{T}) &:= -2\mathcal{T} + \Lambda C F + F^T C^T \Lambda, \end{aligned} \tag{4}$$

then the solution $x(t) \equiv 0$ of the above stochastic system with $D = \sigma I$ is asymptotically stable in probability.

Problem Solution - (contd.)

- Apply the infinitesimal generator of the novel non-quadratic Lyapunov functional, where $\rho = 1 - \nu/2$

$$V(x) = (x^T x)^{\nu/2} + \Sigma_{k=1}^{n} \lambda_k \int_0^{y_k} s^{-2\rho} f_k(s) ds.$$

- Define

$$-\mathcal{F}_0 := V_x^T (Ax + Ff) + \frac{1}{2} x^T D^T V_{xx} Dx \qquad (5)$$

and the nonlinear constraints

$$-\mathcal{F}_i := (x^T x)^{-\rho} f_i(y_i) (f_i(y_i) - s_i y_i) \leq 0$$

Problem Solution - Proof outline

- Using the $\mathcal{S}$-procedure we get that $\mathcal{L}V < 0$ is satisfied with the sector bound constraints if there exist $\tau_1, \ldots, \tau_n \geq 0$ such that $\mathcal{F}_0 - \sum_{i=1}^{n} \tau_i \mathcal{F}_i > 0$.
- Multiplying the resulting inequality by $(x^T x)^\rho$ we readily obtain that

$$\begin{bmatrix} x^T & f^T \end{bmatrix} \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{12}{}^T & \mathcal{N}_{22} \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} < 0$$

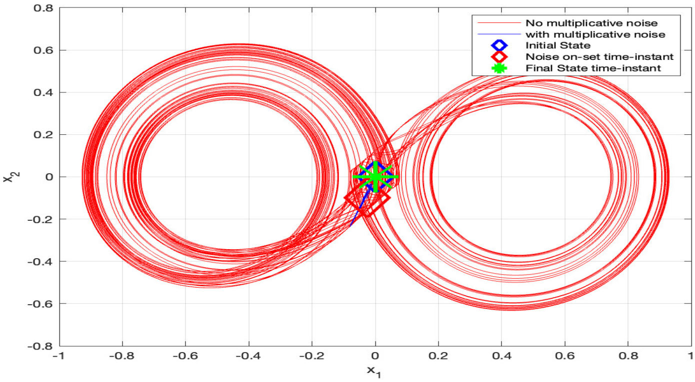
Chaos Control

- Consider a slightly modified version of the 3rd order chaos generator model of [Kwok et. al. 2003].

$$A = \begin{bmatrix} -\epsilon & 1 & 0 \\ 0 & -\epsilon & 1 \\ a_1 & a_2 & a_3 \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, C_1^T = \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}, \quad (6)$$
$$D = \sigma I_3,$$

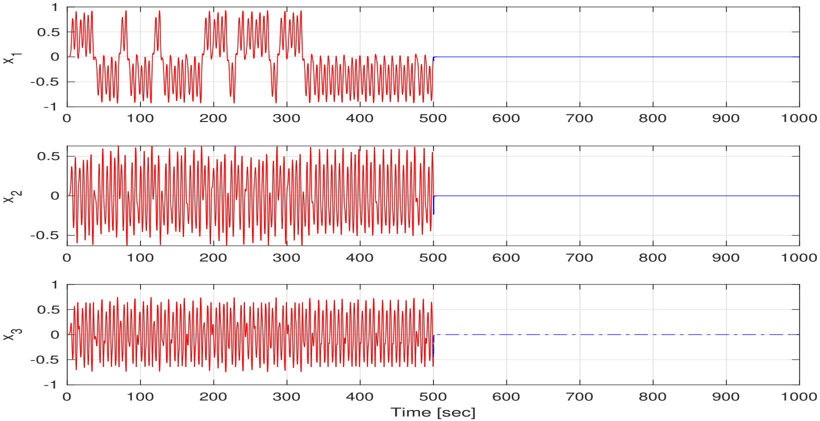
  where $a_1 = -2, a_2 = -1.48, a_3 = -1, \epsilon = 0.01$ and $\beta = 1$.
- The nonlinearity is $f(y_1) = tanh(y_1)$.
- Defining $F = \begin{bmatrix} F_1 & 0_{3\times 2} \end{bmatrix}$, $C = I_3$, $S = diag(1,0,0)$ and $\Delta = diag(1,0,0)$ we obtain feasibility of the LMI condition for $\nu = 0.2$ and $\sigma = 3$.

Chaos Control - Simulation



Apply multiplicative noise at $t \geq 500\,\mathrm{sec}$

Chaos Control - Simulation contd,



Apply multiplicative noise at $t \geq 500\,\mathrm{sec}$

Apply SAR for more General Systems

- We note that the results of the current talk are relevant also in cases where $f(y(t))$ in the model is not a priori sector-bounded.
- In such cases, one may invoke the universal approximation theorem [Cybenko 1989] to systems where a single hidden layer, with, e.g., a *tanh* activation function and a linear output layer, provides an approximation with arbitrarily small error.
- One such example is the Morris-Lecar model of a neuron with three Ion channels and an external applied current I.

Morris Lecar Neuron

Giant Barnacle

Morris Lecar Neuron

- Consider the Morris-Lecar model of a neuron with three Ion channels and an external applied current I.

$$dV/dt = I - g_L(V - V_L) - g_C a M_{ss}(V - V_{Ca}) - g_K N(V - V_K))/C$$

$$dN/dt = (N_{ss} - N)/\tau_N$$

where $M_{ss} = [1 + tanh(V - V_1)/V_2]/2, N_{ss} = [1 + tanh(V - V_3)/V_4]/2$ and

$$\tau_N^{-1} = \phi \frac{cosh(V - V_3)}{2V_4}$$

- V is the voltage across the membrane, N is the recovery variable , C is the capacitance of the cell membrane.

Morris Lecar Neuron (contd.)

- Setting the tuning parameters to $C = 5, V_1 = -1.2, V_3 = 12, V_4 = 17.4, \phi = 1/15, V_L = -60, V_{Ca} = 120, V_K = -80, g_{Ca} = 4, g_K = 8, g_L = 2$ and the initial conditions to $V = -52.14$ and $N = 0.02$ we obtain the oscillatory behavior which is stabilized by injecting a multiplicative noise component in I.
- The different type of behaviors, can be observed in the responses of the three ion channels, L^+, C_a^{2+} and K^+ respectively.

$$L = -g_L(V - V_L)$$

$$Ca = -g_{Ca}M_{ss}(V - V_{Ca})$$

$$K = -g_K N(V - V_K)$$

Morris Lecar Neuron - Approximation

- We define the following functions that are, apparently, non sector bounded :

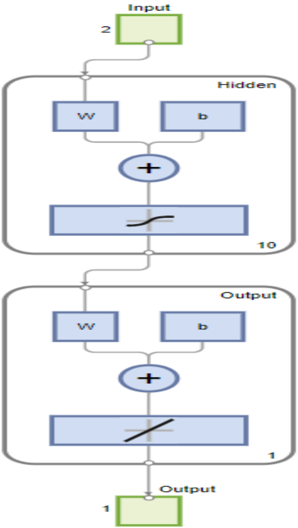
$$f_1 = -g_{C_a} M_{ss}(V - V_{C_a})/C, f_2 = -g_K N(V - V_K)/C$$

$$f_3 = (N_{ss} - N)/\tau_N$$

where we we approximate them, using a shallow neural network that utilizes sector bounded activation functions,

$$f_i(x) = W_2^i(tanh(W_1^i x + b_1^i)) + b_2^i, i = 1, 2, 3$$

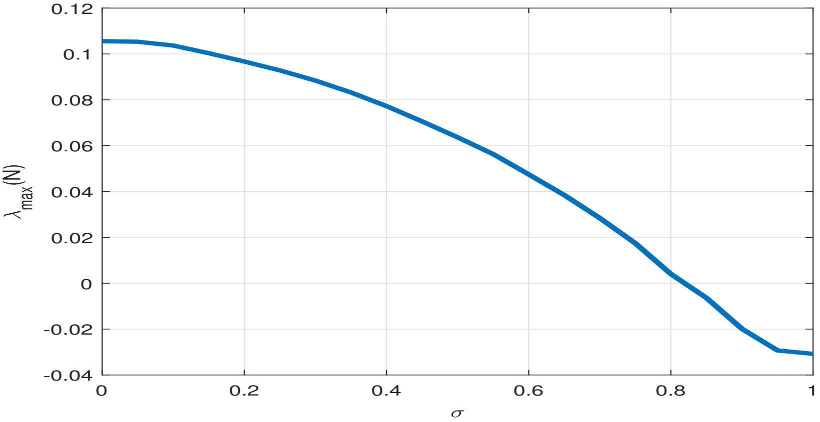
Neural Network Approximation



Morris Lecar Neuron - Analysis and Simulation

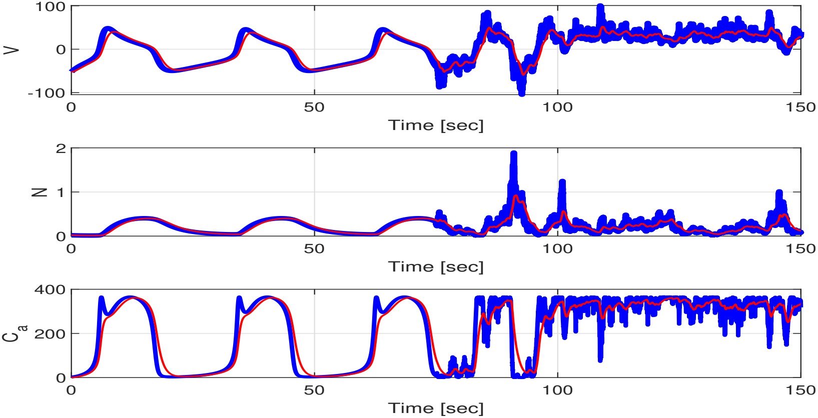
- Each of the f_i are approximated using a network with a single hidden layer and 10 neurons, leading to functions vector f of order 30 and F is of dimension 2×30.
- To fit the resulting model to our original model, we add 28 fictititous states which are zero. To this end, we denote for $\kappa > 0$,
$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & -\kappa I \end{bmatrix}, F = \begin{bmatrix} F \\ 0 \end{bmatrix}$$
- Using this approximation we perform stability analysis, where sweep of SAR level σ values to check the $\mathcal{N} < 0$ condition.
- The simulation results with $\sigma = 0$ and $\sigma = 0.85$ agree with those LMI based stochastic stability analysis results.

Chaos Control - Simulation contd,



Sweep of σ for $\mathcal{N} < 0$ condition

Chaos Control - Simulation contd,



$x_1(t)$ and $x_2(t)$ Moris-Lecar Model - with multiplicative noise

Conclusions

- LMI based conditions for stability of systems with sector bounded nonlinearities have been presented..
- Although the verified type of stability is weaker than other more popular types, interesting stabilization results using multiplicative noise only emerge.
- It seems that SAR can be analyzed and applied also in more general cases, with the use of the universal approximation theorem.